ALGEBRA

NEWCOMB'S
MATHEMATICAL
COURSE

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NEWCOMB'S MATHEMATICAL COURSE

A
SCHOOL ALGEBRA

BY

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PREFACE.

THE guiding rule in the preparation of the present work has been to present the pupil with but one new idea at a time, and, by examples and exercises, to insure its assimilation before passing to another. This system is carried out by a minute subdivision of all the algebraic processes, and the end kept in view is that in no case where it can be avoided shall the pupil have to go through a process of which he has not previously learned all the separate steps. As a part of the same plan, definitions are so far as practicable brought in only as they are wanted, the first exercises in indicated operations are performed with numbers, and the pupil is set to work on exercises from the start.

Correlative with the system of subdivision is that of extending the scope of the exercises so as to include not only the elementary operations of algebra, but their combinations and applications. It is hoped that the wider range of thought and expression in which the pupil is thus practised will be found to tell in his subsequent studies.

As another part of the same general plan the subject has been divided into three separate courses. The First Course, which extends to Simple Equations, is intended to drill the student in all the fundamental processes by exercises which are, for the most part, of the simplest character. The varied exercises in algebraic expression which are scattered through this course form a feature to which the attention of instructors is especially solicited.

In the Second Course the processes are combined and the whole subject is treated on a higher plane. The general arrangement of this course is the same as that of the Elementary Course in the author's *College Algebra.* But the exercises are all different, and greater simplicity of treatment is aimed at. This course terminates with Quadratic Equations.

The Third Course consists of three supplementary chapters which, however, should be mastered before entering college.

An attempt has been made to treat quadratic equations with such fulness as to avoid the usual necessity of reviewing that subject after entering college.

In the preparation and use of such a work no question is more difficult than that of the extent to which rigorous demonstrations of the rules and processes shall be introduced. At one extreme we have the old method, in which the teaching is purely mechanical; at the other, the modern demand that nothing be taught of which the reason is not fully explained to and understood by the pupil. The latter method should of course be preferred, but we meet the insuperable difficulty that we are dealing with a subject of which the reasoning cannot be understood until the pupil is familiar with it. The rule adopted in the present case has been to present a proof, reason or explanation wherever it was thought it could be clearly mastered. Most teachers will, however, admit that long explanations of any kind weary the pupil more than they instruct him, and that the best course is to present the examples and exercises in such a form that their logical correctness shall gradually become evident without much further help.

Were the author to make a suggestion respecting the system of teaching to be adopted, it would be to commence the study of algebra at least one year earlier than usual, and to devote the whole of that year to the first course, taking two or perhaps three short lessons a week. The habit of using algebraic symbols in working and thinking would thus become established before taking up the subject in its more difficult forms.

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FIRST COURSE.

THE ELEMENTARY PROCESSES OF ALGEBRA.

REFERENCE LIST

OF ALGEBRAIC SIGNS AND SYMBOLS.

$+$ *Plus,* sign of addition.

$-$ *Minus,* sign of subtraction.

$\times$ *Times,* sign of multiplication.

$\div$ *Divided by,* sign of division.

$:$ *Is to,* or *divided by,* sign of ratio.

$::$ *So is,* sign of equality of ratios.

$=$ *Equals,* sign of equality.

$\equiv$ *Represents,* sign that $\begin{cases}\text{two expressions are identically equal.}\\ \text{a symbol stands for an expression (§ 109).}\end{cases}$

$>$ *Greater than.* $A > B$ means A is greater than B.

$<$ *Less than.* $A < B$ means A is less than B.

( ) *Parentheses,* } *signs of aggregation,* indicate that the included expression is to be treated as a single quantity.
$\overline{\quad}$ *Vinculum,*

$\sqrt{\ }$ *Radical,* indicates a root.

$\sqrt[n]{\ }$ nth *root.*

. . . . . *Continued,* stand for any number of unwritten letters or terms.

$\because$ *Because.*

$\therefore$ *Therefore,* or *hence.*

$\propto$ *Infinity,* represents a quantity increasing beyond all limits.

SCHOOL ALGEBRA.

CHAPTER I.

THE ALGEBRAIC LANGUAGE.

General Definitions.

1. *Definition.* **Mathematics** is the science which treats of the relations of magnitudes.

2. *Def.* A **magnitude** is that which can be divided into any number of separate parts.

EXAMPLE. Length, breadth, time and weight are magnitudes.

3. *Def.* A **quantity** is a definite portion of any magnitude.

EXAMPLE. Any number of feet, miles, degrees, bushels, years or dollars is a quantity.

4. In arithmetic and algebra quantities are expressed by means of numbers.

To express a quantity by a number we take a certain portion of the magnitude as a unit, and state how many of the units the quantity contains.

EXAMPLE. We may express the length of a string by saying that it is 7 feet long. This means that we take a certain length called a foot as a unit, and that the string is equal to 7 of these units.

Algebraic Signs.

5. *Sign of equality.*

$=$ read *equals*, or *is equal to*, is the sign of equality. It indicates that the quantity which precedes it is equal to the quantity which follows it.

6. *Signs of addition and subtraction.*

$+$ read *plus*, is the sign of addition. It indicates that the quantity which follows it is to be added to the quantity which precedes it.

Def. The number resulting from the addition is called the **sum.**

$-$ read *minus*, is the sign of subtraction. It indicates that the quantity which follows it is to be subtracted from the quantity which precedes it.

Def. The quantity from which we subtract is the **minuend.**

The quantity subtracted is the **subtrahend.**

The result is the **remainder.**

EXAMPLES. The expression

$$9 + 4 = 13$$

means 4 added to 9 makes 13.

$$9 - 4 = 5$$

means 4 subtracted from 9 leaves 5.

REMARK. Numbers to be added or subtracted may be written in any order. Thus $-5 + 7$ is the same as $7 - 5$, so that $-5 + 7 = 7 - 5 = 2$.

EXERCISES.

1. $15 + 7 =$ what?
2. $15 - 7 =$ what?
3. $-7 + 15 =$ what?
4. $-8 + 8 =$ what?

7. *Positive and negative quantities.*

Def. The signs $+$ and $-$ are called the **algebraic signs.**

The sign $+$ is called the **positive sign.**

The sign $-$ is called the **negative sign.**

Def. **Positive quantities** are those which have the sign $+$ before them.

Negative quantities are those which are preceded by the sign $-$.

8. We may have any number of quantities connected by the signs $+$ and $-$.

EXAMPLE. The expression

$$9-2-4+5$$

means 2 to be subtracted from 9, leaving 7; then 4 more to be subtracted, leaving 3; then 5 to be added, making 8.

Hence we may write

$$9-2-4+5+12-2=18.$$

But when we have several quantities connected by the signs $+$ and $-$ it is generally easiest to add all the positive quantities and all the negative quantities separately, and then subtract the sum of the negative from the sum of the positive quantities.

EXAMPLE. The preceding expression is calculated thus:

$$\begin{array}{rcr} 9-2 & & \\ 5-4 & & 26 \\ 12-2 & & -8 \\ \hline 26-8 & & 18 \end{array}$$

We add 9, 5, and 12, making 26. Then we add -2, -4, and -2, making -8. Then $26-8=18$.

EXERCISES.

Compute the values of the following expressions:

1. $14-3-8+22+17.$
2. $41-9-10-11+1.$
3. $12-13+14-15+16.$
4. $1+2+3+4-9.$
5. $1+3+6+10+15.$
6. $1+4-9-16+25.$
7. $1+8+27+64+125.$
8. $1-8+27-64+125.$
9. $24-31+1-2+50.$
10. $9+19+29-39+40.$
11. $-24-13+7-4+101.$

12. $9 - 5 + 5 - 9 + 14.$
13. $-32 - 23 + 50 + 32 + 23.$
14. $117 - 13 - 31 - 17 - 56.$
15. $1008 - 1008 - 500 + 650.$
16. $1205 + 1336 - 296 - 1694.$
17. $439 + 940 - 631 - 142.$

Parentheses.

9. When combinations of numbers are enclosed in parentheses the results to which they lead are to be treated as if they were single numbers or quantities.

EXAMPLE. $19 - (9 - 4)$ means that the quantity $9 - 4$, that is 5, is to be subtracted from 19. The remainder is 14. Therefore

$$19 - (9 - 4) = 19 - 5 = 14.$$

The expression $12 - (7 - 3) + (8 - 2) - (9 - 6)$ means that $7 - 3$, that is 4, is to be subtracted from 12; then $8 - 2$, that is 6, is to be added; then $9 - 6$, that is 3, is to be subtracted.

We may write the expression thus:

$$12 - (7 - 3) + (8 - 2) - (9 - 6) = 12 - 4 + 6 - 3 = 11.$$

EXERCISES.

Compute the values of the following expressions:

1. $12 - (11 - 10).$
2. $17 - (13 - 4).$
3. $17 - (13 + 4).$
4. $46 + (19 - 4).$
5. $46 - (19 - 4).$
6. $13 + (14 + 15).$
7. $27 - (16 - 9 - 4).$
8. $41 - (31 + 1 - 7).$
9. $(101 - 99) + (1 + 8 + 27).$
10. $(34 + 13 - 9) - (34 - 13 - 9).$
11. $1 + 3 + 6 - (1 - 3 + 6).$
12. $(1 + 3 + 6) - 1 - 3 + 6.$
13. $(9 - 8) - (8 - 7) + (7 - 6).$
14. $-(9 - 8) + 18 - (33 - 17).$
15. $(74 - 69) - (8 + 3 - 7) + 145.$

16. $(27 + 8) - (27 - 8) + (1 + 2 - 3) - (1 - 2 + 3)$.
17. $-(11-7)+(134-7-19-6)-27+(1+2-3+4-5)$.
18. $(136 - 48) + (49 - 1 - 35) - (10 - 7 + 13)$.
19. $(7-11+13)-(7-11+13)+(13-11)-(11-13+4)$.
20. $(746 - 614) - (42 - 18 - 17) + 973-39-(973 - 39)$.
21. $8 + 1 - (8 - 1)$.
22. $8 + 2 - (8 - 2)$.

10. Signs of Multiplication.

$\times$ read *multiplied by*, or *times*, indicates that the numbers between which it stands are to be multiplied together.

EXAMPLE 1. $12 \times 5 = 60$ means 12 multiplied by 5 make 60.

Ex. 2. $(11 - 3) \times (4 + 2)$ means that the remainder, when 3 is subtracted from 11, is to be multiplied by the sum $4 + 2$, that is 6. Therefore

$$(11 - 3) \times (4 + 2) = 8 \times 6 = 48.$$

Def. The quantity to be multiplied is called the **multiplicand.**

The number by which it is multiplied is the **multiplier.**

The result is called the **product.**

Multiplier and multiplicand are called **factors.**

11. *Omission of the sign* $\times$. The sign $\times$ is generally omitted, and when two quantities are written after each other without any sign between them it indicates that they are to be multiplied together.

Between numbers a dot may be inserted.

The reason for inserting the dot is that such a product as 2×5 would be mistaken for 25 (*twenty-five*) if nothing were inserted.

EXAMPLES. $2(5 + 3)$ means the sum of 5 and 3 multiplied by 2. Therefore

$$2(5 + 3) = 2 \,.\, 8 = 16.$$

$3 \,.\, 7 \,.\, 2$ means 3 times 7 times 2, which makes 42.

12. We may have any number of quantities multiplied together. Any two of them are then to be

multiplied together; this product multiplied by the third, this product multiplied by the fourth, this product again by the fifth, etc.

EXAMPLES. $2 . 3 . 4 = 2 \times 3 \times 4 = 6 \times 4 = 24.$

$3 . 5 . 2(1 + 3)(1 + 5) = 3 . 5 . 2 . 4 . 6 = 720.$

We may get this result thus: $3 . 5 = 15$; $15 . 2 = 30$; $30 \times 4 = 120$; $120 \times 6 = 720$.

The multiplications may be performed in any order without changing the result.

EXERCISES.

Calculate the values of the following expressions:

1. $4(9 - 2 + 3)$. Ans. 40.
2. $(9 - 6)(6 - 3)$. Ans. 9.
3. $(5 - 2)(10 - 3)$. Ans. 21.
4. $(7 - 4)(17 - 4)$. Ans. 39.
5. $3(13 - 7)(19 - 14)$. Ans. 90.
6. $(8 - 4)4(9 - 5)$. Ans. 64.
7. $(1 - 2 + 3)(4 - 5 + 6)$.
8. $(1 + 2)(2 + 3)(3 + 4)(4 + 5)$.
9. $10(13 - 9)(5 - 1)$.
10. $17(17 - 1)$.
11. $(13 + 7)(13 - 7) = 13 . 13 - 7 . 7$.
12. $(8 + 5)(8 - 5) = 8 . 8 - 5 . 5$.
13. $(19 - 3)(19 - 3)$.
14. $(14 + 4)(14 - 4)$.
15. $(1 + 9)(1 + 10)$.
16. $(3 + 4 + 5)(3 + 4 - 5)$.
17. $(3 + 4 + 5)(3 - 4 + 5)$.
18. $(3 - 4 + 5)(3 + 4 - 5)$.
19. $(25 + 10 + 4)(5 - 2) = 5 . 5 . 5 - 2 . 2 . 2$.
20. $9(9 - 3) - 8(8 - 3) + 7(7 - 3)$.
21. $(8 - 2)(7 - 3) - (6 - 3)(5 - 2)$.
22. $(6 . 8 - 5 . 9)(2 . 3 . 4 - 4 . 5)$.
23. $(7 . 8 - 2 . 12)(5 . 6 - 4 . 7)$.
24. $(2 . 3 . 4 - 20)(4 . 5 . 6 - 7 . 5 . 3)$.
25. $3(2 . 6 - 3 . 1)(4 . 5 - 2 . 8)$.
26. $2(2 . 3 - 3 . 1)(4 . 5 - 2 . 7)5$.

13. Signs of Division.

$\div$ read *divided by*, indicates that the number which precedes it is to be divided by that which follows it.

EXAMPLES. $15 \div 3 = 5$;

$19 \div 7 = \frac{19}{7} = 2\frac{5}{7}$.

Def. The quantity to be divided is called the **dividend.**

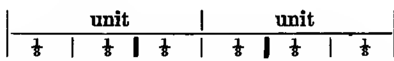
That by which it is divided is the **divisor.**

The result is called the **quotient.**

14. Division in algebra is commonly expressed by writing the divisor below the dividend with a line between them, thus forming a common fraction, which fraction will be the quotient.

Illustration. Let us show that the quotient $2 \div 3 = \frac{2}{3}$.

This means that if we take two units and divide the sum of them into three equal parts, each of these parts will be $\frac{2}{3}$ of the unit.



Let us draw a line as above, two inches in length, the inch being the unit. Divide each unit into three equal parts. Each of these parts will then be $\frac{1}{3}$. Since the two units have 6 parts in all, one third of the line will be 2 parts, that is $\frac{2}{3}$.

EXERCISES.

1. Show in the same way that if we divide a line 3 units long into 5 equal parts, each part will be $\frac{3}{5}$ of the unit.

We first divide each unit into 5 parts, making 15 fifths. Dividing these 15 parts by 5 we shall have 3 parts, that is $\frac{3}{5}$, for the quotient.

2. Divide a line 5 units long into 3 parts, and show that each part is $\frac{5}{3}$ of the unit.

We divide each unit into thirds, making 15 thirds in all.

3. Divide a line 7 units long into 4 parts, and show that each part is $\frac{7}{4}$ of the unit.

4. $\dfrac{9 + 20 - 2}{7 - 4} = \dfrac{27}{3} = 9.$

5. $\dfrac{2 \cdot 3\,(6+8)}{(3+4)\,(5-3)} = \dfrac{2 \cdot 3 \cdot 14}{7 \cdot 2} = 6.$

6. $\dfrac{3+12-(9-7)}{2 \cdot 3} = \dfrac{15-2}{6} = \dfrac{13}{6} = 2\frac{1}{6}.$

7. $7 \times \dfrac{8-2+9}{8-4} = 7 \times \dfrac{15}{4} = \dfrac{105}{4} = 26\frac{1}{4}.$

NOTE. We obtain this result by multiplying the numerator of the fraction by the multiplier 7. It is explained in arithmetic that we multiply a fraction by multiplying its numerator.

8. $\dfrac{13-4+9}{(1+2)\,2}.$

9. $\dfrac{14-(14-4)+17}{1+2+4}.$

10. $9 \times \dfrac{(2+1)\,(2+1)}{(1+2)\,(1+2-3+4)}.$

11. $\dfrac{(8+1)\,(1+8)\,8+1}{8-1}.$

12. $2 \cdot \dfrac{1+(1+2)\,2+(1+2+3)\,3}{2\,(4-3+2-1)+3\,(3-2+1)}.$

13. $\dfrac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8}.$

14. $\dfrac{1+7 \cdot 4\,(7+4)\,(7-4)}{2\,(4+7)+7}.$

15. $\dfrac{8 \cdot 7 \cdot 6 \cdot 5}{1 \cdot 2 \cdot 3 \cdot 4} \div \dfrac{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}.$

16. $\dfrac{(2 \cdot 2+3)\,(3 \cdot 3-2)}{7 \times 7}.$

17. $\dfrac{13 \cdot 13+31}{2 \cdot 5 \cdot 5 \cdot 2}.$

18. $\dfrac{(1 \cdot 3)+(3 \cdot 5)+(5 \cdot 7)}{(2 \cdot 4)+(4 \cdot 6)+(6 \cdot 8)}.$

19. $\dfrac{(1 \cdot 1 \cdot 1)+(2 \cdot 2 \cdot 2)+(3 \cdot 3 \cdot 3)+(4 \cdot 4 \cdot 4)}{(1+2+3+4) \times (1+2+3+4)}.$

20. $2 \times \dfrac{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6}{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 7}.$

Symbols of Quantity.

15. In algebra quantities are represented by the letters of the alphabet.

Def. The letters used to represent quantities are called **symbols**.

16. A quantity may be represented by any symbol we choose. The symbol may then be regarded as a *name* for the quantity.

EXAMPLE. We may draw a straight line and call it a. The letter a is then the name of that line. But we might also have called it b, c, x, or any other name we choose. The only restriction is that two quantities must not have the same symbol or name in the same question.

17. *Def.* The **value** of a symbol is the number or quantity which it represents.

EXAMPLE. In the preceding example the length of the line is the value of the symbol a.

If I have the number 275 and call it n, then 275 is the value of n.

Reduction of Algebraic Expressions to Numbers.

18. *Def.* An algebraic **expression** is any combination of algebraic symbols.

RULE. *To reduce an algebraic expression to numbers we put, in place of the symbols, the numbers which they represent, and perform the operations indicated.*

EXAMPLE. Find the value of the following expression,

$$\frac{cn - ab}{3n - 6c},$$

supposing

$$\begin{array}{ll} a = 3, & g = 12, \\ b = 6, & n = 15, \\ c = 7, & m = 15. \end{array}$$

We do this as follows:

$cn = 7 \cdot 15 = 105;$ $\quad 3n = 3 \cdot 15 = 45;$

$ab = 3 \cdot 6 = 18;$ $\quad 6c = 6 \cdot 7 = 42;$

hence $cn - ab = 87;$ hence $3n - 6c = 3;$

and then $\frac{87}{3} = 29.$ Ans.

EXERCISES.

Using the above values of the symbols a, b, c, g and n, compute the values of:

1. $a + b + c.$
2. $a + b + c - g.$
3. $n - g - a.$
4. $gc + ab.$
5. $gc - ab.$
6. $an - (a + b + c).$
7. $abc - cg.$
8. $gn + cg - bc + ab.$
9. $3ab - 4bc + 5cgn.$
10. $2abc - 3ab + 5a.$
11. $\dfrac{cn + ab}{cn - ab}.$
12. $\dfrac{bn + cg}{bn - cg}.$
13. $\dfrac{aaa + bb - gc}{2ng}.$
14. $\dfrac{3aaaa - 2bbb + aabb}{aa + bb - 9}.$
15. $\dfrac{\frac{g}{a} + \frac{2n}{b}}{\frac{g}{a} - \frac{2n}{b}}.$
16. $\dfrac{\frac{gn}{a + b} + \frac{n}{a}}{\frac{g}{a} + 1}.$

If $a = 1$, $b = 2$, $c = 3$, $d = 4$, $e = 5$, etc., then find the values of:

17. ab; abc; $abcd$; $abcde$.
18. $ab + abc + abcd + abcde.$
19. $ab\,[1 + c + cd\,(1 + e)].$
20. $\dfrac{a}{b} + \dfrac{c}{f} + 2\dfrac{a + c}{b + f}.$
21. $\dfrac{b + a}{b - a} \times \dfrac{d + c}{d - c}.$
22. $(a + b)\,(c + d)\,(e + f).$
23. $a + b\,(c + d)\,(e + f).$
24. $(a + b)\,c + d\,(e + f).$
25. $(a + b + c + d)\,e + f.$
26. $\dfrac{(a + b)\,b + (b + c)c}{(b + c)c + (c + d)d - 1}.$
27. $\frac{1}{2}f\,(f - 1).$
28. $\dfrac{abc}{def}.$
29. $\dfrac{ace}{bdf}.$
30. $(f - e)\,(e - d)\,(d - c)\,(c + d + e + f).$
31. $\dfrac{7b - 4a}{4b - 7a} - \dfrac{6d - 5c}{5d - 6c}.$
32. $\dfrac{2a + b}{2b + a}\{1 + c\,(1 + d)\}.$
33. $a\{1 + b\,[2 + c\,(3 + d)]\}.$

34. Find the values of the following expression, for $n = 0$, then for $n = 1$, then for $n = 2$, etc., to $n = 7$:

$$\frac{n(n+1)(n+2)}{6}.$$

For $n = 0$, the expr. $= 0$; for $n = 1$, the expr. $= 1$; for $n = 2$, the expr. $= 4$, etc.

When	$n = 0$,	the expression	$=$	0.
"	$n = 1$,	" "	$=$	1.
"	$n = 2$,	" "	$=$	4.
"	$n = 3$,	" "	$=$	10.
"	$n = 4$,	" "	$=$	20.
"	$n = 5$,	" "	$=$	35.
"	$n = 6$,	" "	$=$	56.
"	$n = 7$,	" "	$=$	84.

In like manner find the values of:

35. $\frac{1}{2}n(n+1)$, for $n = 0, 1, 2, 3, 4$, and 5.

36. $\frac{1}{6}n(n+1)(2n+1)$, for same values.

37. $\frac{1}{4}n \,.\, n(n+1)(n+1)$, for same values.

38. $\dfrac{n}{2n+1}$, for same values.

39. Can you show that the quantity $8 + n + (8 - n)$ is the same for all values of n?

Powers and Exponents.

19. *Def.* A **power** of a quantity is the product obtained by taking the quantity as a factor any number of times.

Def. The **degree** of a power means the number of times the quantity is taken as a factor.

If a number is to be raised to a power, the result may, in accordance with the rule for multiplication, be expressed by writing the number the required number of times.

EXAMPLES. $5 \,.\, 5 = 25$ is the second power of 5.
bbb is the third power of b.
$xxxxx$ is the fifth power of x.

To save repetition, the number whose power is to be expressed is written only once, and the number of times it is taken as a factor is indicated by small figures written after and above it.

EXAMPLES. Instead of $5 . 5$ we write 5^2.
" " aa " " a^3.
" " $bbbbb$ " " b^5.

Also, $(1 + 3)^3$ means $(1 + 3)(1 + 3)(1 + 3) = 4 . 4 . 4 = 64$.
$(2 . 3)^2$ " $2 . 3 . 2 . 3 = 6 . 6 = 36$.

20. *Def.* The number which indicates a power is called the **exponent** of that power.

EXERCISES.

Find the values of:

1. 2^4; 4^2; 2^3; 3^2; 5^2; 4^5.

2. $(1 + 5)^3$; $\dfrac{(2 + 5)^2}{7^2}$; $\dfrac{2^3(2 + 3)^3}{10}$; $(1 + 2)^{1+2}$.

3. $2^2 . 3^2$; $(1 + 2)^2 (2 + 3)$;
$(1 + 2) (2 + 3)^2 [(1 + 2) (2 + 3)]^2$.

4. $1 + 4^2$; $(1 + 4)^2$; $(7 - 5)^{7-6}$; $7^2 - 5^{7-6}$;
$(10 - 7) (9 - 6) (8 - 5) = 3^3$.

5. $\dfrac{(10 + 5)^2}{5}$; $\dfrac{10 + 5^2}{5}$; $\dfrac{10^2 + 2 . 5 . 10 + 5^2}{5}$.

6. $1 + 2 + 3^3$; $1 + (2 + 3)^3$; $(1 + 2 + 3)^2$.

7. $\dfrac{1^3 + 2^3 + 3^3 + 4^3 + 5^3}{(1 + 2 + 3 + 4 + 5)^2}$.

8. $\dfrac{1 + 2 + 2^2 + 2^3 + 2^4 + 2^5}{2^6 - 1}$.

9. $\dfrac{1 + 3 + 3^2 + 3^3 + 3^4 + 3^5}{3^6 - 1}$.

10. $\dfrac{2^4 . 4^2 - 200}{7 . 2^3}$; $7 . 2^3$; $(7 . 2)^3$; $7^3 . 2$.

11. $\dfrac{(4 + 7)^2 + 11(7 + 4)^3 - (13 - 2)}{(19 - 9)^3 + 2^5 - 5^2 - 7}$.

12. $(6 - 5)^3 + (6 - 4)^3 + (6 - 3)^3$
$+ (6 - 2)^3 + (6 - 1)^3 = \dfrac{5^2(5 + 1)^2}{2^2}$.

13. $\dfrac{7(8 - 1) (9 - 2) (10 - 3) (11 - 4)}{7^7}$.

14. $87 - 3 . 5^2$; $(87 - 3 . 5)^2$; $87 - 3^2 . 5$.

Write the following expressions with exponents:

15. $mmmm$.
16. $ppppp$.
17. $(a+b)(a+b)(a+b)$.
18. $(a-b)(a+b)(a+b)$.
19. $xxyyzzyzz$.
20. $nn(m+n)(m+n)$.
21. $(p+q)(q+p)rrr$.
22. $tt(x-y)(x-y)(x-y)$.
23. $aaabx(a+x)(a+x)$.
24. $jkk(j+k)(k+j)j$.
25. $(a+b+c)(b+c+a)(c+a+b)$.

Find the values of the following expressions when $a=5$, $b=8$, $m=2$, $n=3$:

26. $(b-a)^2m^2$.
27. $(b-a)^m$.
28. $(b-m)^{b-a}$.
29. $(b-n)^{n-m}$.
30. $(a-n)^{2n}$.
31. $(m+n)^{m+n}$.
32. $m^n n^m$.
33. $\dfrac{(a+m)^m}{(a-m)^n}$.
34. $\dfrac{(b-m)^n}{b^m}$.

Coefficients.

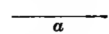
21. *Def.* The coefficient of an algebraic symbol is any number which multiplies it.

EXAMPLE. In the expression

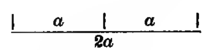
$$3x - 4y + 12z,$$

3 is the coefficient of x; 4 is the coefficient of y; 12 is the coefficient of z.

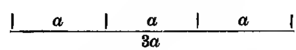
Use of Coefficients. If we call this line a,



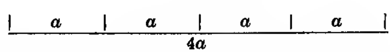
then this line $= 2a$,



this line $= 3a$,



and this line $= 4a$.



REMARK. Compare this definition of coefficient with that of exponent of a power.

$a + a + a + a = 4a$, and 4 is here the coefficient;
also, $a \times a \times a \times a = a^4$, and 4 is here the exponent.

22. A coefficient may be an algebraic symbol or a product of several symbols.

Any quantity may be supposed to have the coefficient 1.

EXAMPLES. In the expression mx, m is the coefficient of x. In the expression $2max$,

$2ma$ is the coefficient of x;
$2m$ is the coefficient of ax;
and 2 is the coefficient of max.

EXERCISES.

In the expression

$$2amx + bcyz + mpqr,$$

What is the coefficient of x?
What is the coefficient of mx?
What is the coefficient of z?
What is the coefficient of yz?
What is the coefficient of mp?

23. Exercises in Algebraic Expression.

NOTE. The object of the following exercises is to practise the student in algebraic expression. The answers are therefore to be expressed or indicated in the manner of algebra, instead of being calculated.

1. How many dollars will 7 knives cost at \$2 each?

To get the cost we must multiply 2 by 7; this is done algebraically by writing $7 \cdot 2$; therefore the required cost is $7 \cdot 2$ dollars.

2. What will 7 pounds of tea cost at \$2 per pound?

3. If I buy 9 sheep at \$5 each, and 15 turkeys at \$2 each, what is the total cost?

4. A man had \$15, and spent \$3; how many dollars had he left?

5. If a boy has a cents in one pocket and b cents in another, how many has he in both? Ans. $a + b$.

6. A man had x dollars, and paid out y dollars; how many had he left?

7. If I buy 7 cows at \$45 each, and 3 horses at \$175, express the total cost? Ans. $\$(7 \cdot 45 + 3 \cdot 175)$.

8. A huckster sold a chickens at x cents each, and lost m cents on his way home; how many cents had he left?

9. A train having to run a distance of m miles, went for 2 hours at the rate of 30 miles an hour; how much farther had it to go?

10. A man bought two pounds of tea for x cents; how much would one pound cost?

Ans. $\frac{x}{2}$ cents.

11. If 8 pounds of tea cost $\$x$, what is the cost per pound?

Ans. $\$\frac{x}{8}$.

12. Three men took dinner at a hotel at a cost of m dollars for all three; how much had each to pay?

13. One boy has m cents and another n cents; if they make an equal division of the money, how much will each have?

14. A boy had b cents, and out of them gave c cents to one person and d cents to another; how many had he left?

15. A quantity m is to be multiplied by another quantity n; express the product.

16. What will $a + b$ sheep cost at x dollars apiece?

17. What will $m + n$ pounds of beef cost at $x + y$ cents a pound?

18. A man bought p dollars' worth of flour at q dollars a barrel; how many barrels were there?

19. How many cents are there in x dollars? How many dollars in x cents?

20. A man had x dollars in one pocket and x cents in another; how many dollars had he in both? How many cents?

21. A man went out with k dollars in his pockets, and paid out m cents; how many cents had he left?

22. What will be the value of h houses at x dollars apiece?

23. How many dollars will m pounds of beef cost at s cents per pound?

24. A man bought from his grocer a pounds of tea at x cents per pound, and b pounds of sugar at y cents per pound, and c pounds of coffee at z cents per pound; how many cents would the bill amount to? How many dollars?

25. A man bought f pounds of flour at m cents per pound, and handed the grocer an x dollar bill to be changed; how many cents ought he to receive in change? How many dollars?

26. Of 3 boys one had n cents in each of his two pockets, another had p cents in each of his three pockets, and the

third had q cents; if they make an equal division of the money, how much would each have?

27. A huckster had 3 large baskets, each containing p apples, and 7 small baskets, each containing q apples. He divided his apples equally among the 10 baskets; how many were there in each basket?

28. The sum of the quantities 17 and 29 is to be divided by the sum of the quantities 5 and 3; what will be the quotient?

29. The sum of the quantities a and b is to be divided by the sum of the quantities m and n; what will be the quotient?

30. A man left to his children a bonds worth x dollars each and s acres of land worth y dollars an acre; but he owed m dollars to each of q creditors; what was the value of the estate?

31. Two numbers x and y are to be added together, their sum multiplied by s, and the product divided by $a + b$; express the quotient. Ans. $\dfrac{s(x+y)}{a+b}$.

32. The sum of the numbers p and q is to be divided by the sum of the numbers a and b, forming one quotient, and the difference of the numbers p and q is to be divided by the difference of the numbers a and b, forming another quotient; express the sum of the two quotients.

33. In the preceding exercise express the product resulting from multiplying the sum of the quotients by m.

34. The quotient of x divided by a is to be subtracted from the quotient of y divided by b, and the remainder multiplied by the sum of x and y divided by the difference between x and y; express these operations in algebraic language.

35. The number x is to be increased by n, the sum is to multiplied by $a + b$, and q is to be added to the product, and the sum is to be divided by rs; express the result.

36. The quotient of x divided by y is to be divided by the quotient of a divided by b.

37. The quotient of x divided by y is to be added to the quotient of a divided by b, and the sum is to be divided by the sum of m and n.

38. $x + y$ houses each had $a + b$ rooms, and each room $m + n$ pieces of furniture; how many pieces were there in all?

CHAPTER II.

ALGEBRAIC OPERATIONS.

SECTION I. DEFINITIONS.

24. *Term.* The **terms** of an expression are the parts connected by the signs $+$ and $-$.

EXAMPLE. In the expression $a + 2b - 3c$ the parts a, $2b$ and $3c$ are the terms, there being three terms in all.

Algebraic expressions are divided into *monomials* and *polynomials*.

A **monomial** consists of a single term.

A **polynomial** consists of more than one term.

A **binomial** is a *polynomial* of two terms.

A **trinomial** is a *polynomial* of three terms.

EXAMPLES. The expression $24abx$ is a monomial.

$2ab + 3mxy$ is a binomial.

$l + m + n$ and $x^2 + 2axy + y^2$ are trinomials.

Factor. When any number of quantities are multiplied to form a product, each of them is called a **factor** of the product.

Equation. An **equation** is composed of two equal expressions with the symbol $=$ between them.

EXAMPLE. $7 + 5 = 15 - 3$ is an equation.

Like Terms. Like terms are those which contain identical symbols and differ only in their numerical coefficients.

EXAMPLE. a, $2a$ and $7a$ are like terms; m^2 and $4m^2$ are like; $9x + 12x$ are also like; but $7pq$ and $6qr$ are unlike.

SECTION II. ADDITION AND SUBTRACTION.

Addition of Positive Quantities.

25. The addition of any number of terms may be indicated by writing them down with the sign $+$ between them.

EXAMPLE. The sum of the quantities $2x$, $4ax$, $5b$ and $5bx$ may be written

$$2x + 4ax + 5b + 5bx.$$

If the terms are all *unlike*, this is the only way of expressing addition of algebraic quantities.

26. RULE FOR LIKE POSITIVE TERMS. *If like positive terms are to be added, take the sum of all the coefficients of the symbol and affix the symbol to their sum.*

In this addition a symbol without a coefficient must be considered as having the coefficient 1.

EXAMPLE 1. If we have to add together the quantities x, $2x$, $4x$, $8x$, $12x$, we proceed thus:

$$\begin{array}{r r} & 1x \\ & 2x \\ & 4x \\ & 8x \\ & 12x \\ \hline \text{Sum} = & 27x \end{array}$$

Ex. 2. $x + x + x + x + x = 5x.$
Ex. 3. $am + 2am + 3am = 6am.$

EXERCISES.

Add up the quantities:

1. $9a + 7a + 5a + 3a.$
2. $x + 2x + 3x + 4x.$
3. $2y + 3y + 5y.$
4. $9ab + 9ab + ab.$
5. $ab + ab + ab + ab.$

When there are several sets of similar terms, add each set, and connect the sums by the sign $+$.

EXAMPLE. Add

$$a + 2x + 3a + 7ab + 6x + 5ab + x + 12a.$$

We pick out all the symbols a and write them under each other with their coefficients; then all the symbols x; then all the products ab. We then add the coefficients, thus forming the sum.

Work:

$$\begin{array}{r r r r} & +\,1a & +\,2x & +\,7ab \\ & 3a & +\,6x & +\,5ab \\ & 12a & +\,1x & \\ \hline \text{Sum} = & 16a & +\,9x & +\,12ab \end{array}$$

6. Add
$$\begin{array}{r} 12a + 4b + 19abc \\ a + 2b + 30abc \\ 3a + 6b + 12abc \\ \hline \end{array}$$

7.
$$\begin{array}{l} ab + 3e \\ 22ab + 12e + d \\ 29ab + e \\ \hline \end{array}$$

8. $(x + y) + 7(x + y) + 8(x + y)$. Ans. $16(x + y)$.

9. $(a - b) + 2(a - b) + 3(a - b)$.

10. $(m - p) + (m - p) + 3(m - p) + 24(m - p)$.

11. $3(m - n)x + 5(m - n)x + 7(m - n)x$.

12. If Thomas has x dollars, John 3 times as many as Thomas, and James as many as John and Thomas together, express the number all three have in the language of algebra.

13. If a man owes his grocer b dollars, his tailor c dollars, his shoemaker 4 times as much as his grocer, and his butcher twice as much as his shoemaker, what is his total indebtedness?

Addition of Negative Quantities.

27. Let us have to add $7 - 4$, that is 3, to 10. This is the same thing as adding 7 and subtracting 4, because $10 + 7 - 4$ is the same as $10 + 3$, namely, 13.

Whatsoever numbers a, b and c represent, if we add $b - c$ to a the sum will be $a + b - c$.

Hence

RULE. *If negative quantities are among those to be added, they are to be subtracted from the sum of the others.*

Algebraic addition therefore means something more than addition in arithmetic or arithmetical addition, because it may include subtraction.

28. *Def.* **Algebraic addition** means the combination of quantities according to their algebraic signs, the positive ones being added, and the negative ones subtracted.

Def. The result of algebraic addition is called the **algebraic sum.**

EXAMPLE. The algebraic sum of $-2 -3 + 15 - 6$ is 4.

Def. **Numerical addition** and **numerical subtraction** mean addition and subtraction as in arithmetic, without regard to the algebraic signs of the quantities.

29. Rule for Algebraic Addition. *Take the sum of all the positive terms and the sum of all the negative terms. Subtract the less sum from the greater and, if the negative sum is the greater, write the sign — before the difference.*

Remark. If the positive and negative sums are equal, the total sum will be zero.

Example 1. $4 - 2 + 3 - 8 = +7 - 10 = -3.$

Here the sum of the positive quantities is 7 and of the negative ones 10. In arithmetic we cannot subtract 10 from 7, but in algebra we express this subtraction by taking 7 from 10 and writing — before the difference to indicate that the subtractive quantities are the greater.

Ex. 2. Add

$$\begin{array}{rrr} 4x & +\ 2ay & -\ 3z \\ 3x & -\ 4ay & -\ \ z \\ 2x & -\ 3ay & +\ 4z \\ \hline \text{Sum, } 9x & -\ 5ay & \end{array}$$

Ex. 3. Add

$$\begin{array}{rrr} 7x & -\ 5y & +\ 4 \\ -\ 2x & -\ 3y & -\ 8 \\ -\ 5x & +\ 8y & +\ 4 \\ \hline \text{Sum, } 0 & 0 & 0 \end{array}$$

EXERCISES.

Add:

1. $3x - 4x + a + 7x + 5x + 2b + 3a - 7x - 5a.$
2. $2ay - 2y + 6y + 3ay - 7y + 10xy + 6ay - 5xy.$
3. $8x + 9 - 3y + 7x - 12 + 8y - 2x - 3 - 7y.$
4. $7y + 6y + 3a - 9y - 2y - 5a + 2y - 4y + 2a.$
5. $9x + axy + 3y - 5x - 2xy + 6y.$
6. $9x - 3xy - 10x + 4xy - 4x - 6x.$
7. $4a - 2b, \quad 6a - 5b, \quad 8a - 11b, \quad a + 7b.$
8. $4x - 3y, \quad 3x - 5y, \quad -x + y, \quad -6x + 4y.$
9. $5a + 3b + c, \quad 3a + 3b + 3c, \quad a + 3b + 5c.$
10. $3x + 2y - z, \quad 2x - 2y + 2z, \quad -x + 2y + 3z.$
11. $7a - 4b + c, \quad 6a + 3b - 5c, \quad -12a + 4c.$
12. $x - 4a + b, \quad 3x + 2b, \quad a - x - 5b.$
13. $a + b - c, \quad b + c - a, \quad c + a - b, \quad a + b - c.$
14. $a + 2b + 3c, \quad 2a - b - 2c, \quad b - a - c, \quad c - a - b.$
15. $a - 2b + 3c - 4d, \quad 3b - 4c + 5d - 2a, \quad 5c - 6d + 3a - 4b.$
16. $a - 2b + c, \quad 2a - 3b - 4c, \quad 3c - 4d.$
17. $2x + 3, \quad 5x + 7, \quad 13x + 1, \quad x - 4.$
18. $ax + b, \quad cx + d, \quad ex + f.$
19. $my + n, \quad 5ay - n, \quad 2n - 7ay.$
20. $a + b + c, \quad a + b - c, \quad a - b + c, \quad -a + b + c.$
21. $x^2 + 2xy + y^2, \quad x^2 - 2xy + y^2, \quad 2x^2 - 2y^2.$
22. $x^2 - y^2, \quad y^2 - z^2, \quad z^2 - x^2.$

23. $17ax - 19b$, $13x - c$, $11b - a + c$, $8b - 10ax$.
24. $7a - 8b + 9c$, $9b - 7c - 8a$, $9a - 7b + 8c$.
25. $a^3 + 1$, $a^2 + 1$, $a + 1$, $1 - a$, $1 - a^2$.
26. $a + m$, $n + 2a$, $p - 3a$, $4a - q$.
27. $axy + px^2 + my + k$, $4px^2 - 3axy + 9k - 3my$, $14axy - 5px^2 - k$.
28. $m - 2n$, $-2m + 3n - 4p$, $3m - 4n + 5p - 6q$, $-4m + 5n - 6p + 7q - 8r$.
29. $ax^2 + by^2 - a^2b^2$, $ax^2 - by^2 + a^2b^2$, $-a^2x + m - n$.
30. $a - 3y + 2z$, $4a + 7y - 3z$, $y + z - 5a$.

Subtraction.

30. *Def.* **Subtraction** consists in expressing the difference between two algebraic quantities.

EXAMPLE 1. Let us have to subtract $5 - 2$ from 11. Since $5 - 2 = 3$, we must subtract 3 from 11, leaving 8. But this is the same thing as first subtracting 5 and adding 2. Hence the result is

$$11 - 5 + 2,$$

and the sign of 2 is changed from $-$ to $+$.

EX. 2. Take $a - b$ from y.

It is plain that if we subtract a from y we shall subtract b too much, because $a - b$ is less than a. But if after subtracting a we add b, we shall make the answer right. Hence the answer is

$$y - a + b.$$

In this answer the positive symbol a has the sign $-$ in the remainder and the negative symbol b the sign $+$.

We thus derive the following rule for subtraction:

31. RULE. *Change the signs of all the terms to be subtracted, or imagine them to be changed, and then proceed as in addition.*

NUMERICAL EXAMPLES.

The learner should first practice on the purely numerical exercises until he sees how the rule leads to correct results. Each result should be proved by showing that the answer is right.

$$\begin{array}{ll} \text{From} & 21 + 10 \\ \text{Take} & 9 - 5 \\ \hline \text{Rem.} & 12 + 15. \end{array}$$

Operation. Imagine 9 to have the sign minus. We must by § 29 subtract it from 21, leaving 12. Imagining 5 to have the sign +, we must add it to 10, making + 15. So the answer is 12 + 15.

Proof.

$$\begin{array}{lr} \text{Minuend} & = 21 + 10 = 31 \\ \text{Subtrahend} & = \;\;9 - \;\;5 = \;\;4 \\ \hline \text{Remainder} & = 12 + 15 = 27, \end{array}$$

which is right.

$$\begin{array}{llllll} \text{From} & 10 + 6 & & 10 + 6 & 10 + \;\;6 & 10 + \;\;6 \\ \text{Take} & \;\;9 & & \;\;9 - 2 & \;\;9 - \;\;4 & \;\;9 - \;\;6 \\ \hline \text{Rem.} & \;\;1 + 6 & & \;\;1 + 8 & \;\;1 + 10 & \;\;1 + 12 \end{array}$$

$$\begin{array}{llll} \text{From} & 25 - 3 & 27 - 5 & 29 - 7 \\ \text{Take} & 16 + 1 & 15 + 2 & 14 + 3 \\ \hline \text{Rem.} & \;\;9 - 4 & 12 - 7 & \end{array}$$

$$\begin{array}{lll} 31 - 9 & 33 - 4 & 35 - 13 \\ 13 + 4 & 12 + 5 & 11 + 6 \\ \hline \end{array}$$

ALGEBRAIC EXAMPLES.

From $3x - 4ay + 5b + c$

Subtract $x - 7ay + 8b + d$.

We write the minuend, $\qquad 3x - 4ay + 5b + c$

and subtrahend, with signs changed, $- x + 7ay - 8b - d$

Result, applying Rule § 29, $\qquad 2x + 3ay - 3b + c - d$

From $a + b$ subtract $b + y$.

$$\begin{array}{r} a + b \\ - b - y \\ \hline a - y \end{array}$$

EXERCISES.

1. From $\quad 7x - \;\;4bxy - 12cy + \;\;8b + 3ac$

   Take $\quad 2x + \;\;7bxy - \;\;8cy - \;\;5b - 2d$

   Difference, $5x - 11bxy - \;\;4cy + 13b + 3ac + 2d$

NOTE. After the beginner is able to operate by simply imagining the signs changed, he need not actually change them.

2. From $\;\;8a + 9b - 12c - 18d - 4bx + 3cxy$

   Take $19a - 7b - \;\;8c - 25d + 3x \;\; - 4y$

3. From $257z + 201z^2 + 92y + 35ax - 6$
Take $\underline{140z - 82z^2 + 20y + 92ax + 14}$

4. From $8a + 14b$ subtract $6a + 10b$.
5. From $7a - 3b - c$ subtract $2a - 3b - 3c$.
6. From $8a - 2b + 3c$ subtract $4a - 6b - c - 2d$.
7. From $2x^2 - 8x - 1$ subtract $5x^2 - 6x + 3$.
8. From $4x^4 - 3x^3 - 2x^2 - 7x + 9$ subtract $x^4 - 2x^3 - 2x^2 + 7x - 9$.
9. From $2x^2 - 2ax + 3a^2$ subtract $x^2 - ax + a^2$.
10. From $x^2 - 3xy - y^2 + yz - 2z^2$ subtract $x^2 + 2xy + 5xz - 3y^2 - 2z^2$.
11. From $5x^2 + 6xy - 12xz - 4y^2 - 7yz - 5z^2$ subtract $2x^2 - 7xy + 4xz - 3y^2 + 6yz - 5z^2$.
12. From $a^3 - 3a^2b + 3ab^2 - b^3$ subtract $-3ab^2 + b^3$.
13. From $7x^3 - 2x^2 + 2x + 2$ subtract $4x^3 - 2x^2 - 2x - 14$.
14. From $5a^2b - 6mn + 19xy$ subtract $7a^2b + 8mn + 10xy - c$.
15. From $8(a+b) + 12(m+n)$ subtract $3(a+b) + 5(m+n)$.
16. From $9(m+n) - 6(p+q)$ subtract $5(m+n) - 8(p+q) + 3(x+y)$. Ans. $4(m+n) + 2(p+q) - 3(x+y)$.
17. From $9a + 12y + 15(a+y)$ subtract $12(a+y) - 6y + 10a$.
18. From $19\frac{a}{b} + 23\frac{c}{d} - 18\frac{x}{y} + k$ subtract $19\frac{a}{b} - 23\frac{c}{d} + \frac{x}{y} - k$.
19. From $a + b$ subtract $a - b$.
20. From $x + 2y$ subtract $x - 2y$.
21. From $2x - 2y$ subtract $x + 2y$.
22. From $a + b$ subtract $b + a$.
23. From $2p + x$ subtract $x + 2p$.
24. From $m + n + x$ subtract $m - n + x$.
25. From $a - b$ subtract $b - a$.
26. From $a + b - c$ take $a - b + c$.
27. From $a - b + c$ take $a + b - c$.
28. From $1 - m + m^2$ take $1 + m + m^2$.
29. From $a^2 - ab + b^2$ take $b^2 - 3ab + a^2$.
30. From $z^2 + z + mn$ take $z - z^2 + nm$.
31. From ab take ba; from $4 . 7$ take $7 . 4$.

Clearing of Parentheses.

32. *Plus Sign before Parentheses.* If a polynomial is enclosed between parentheses and preceded by the sign $+$, the parentheses may be removed and all the terms added without change.

EXAMPLE. $22 + (8 - 15 + 12)$ is the same as $22 + 8 - 15 + 12$.

Proof. $22 + (8 - 15 + 12) = 22 + 5 = 27$
and $22 + 8 - 15 + 12 = 42 - 15 = 27$.

EXERCISES.

Clear of parentheses and add:

1. $5a - 3b + 2c + (-2a + 9b - 3c) + (7a - 16b)$.
2. $36m + p - q + (24m - 5q) + (-42m + 8p)$.
3. $a + b + (a + b - c) + (a - b + c)$.
4. $a + b - c + (a - b + c) + (-a + b + c)$.
5. $a - b + (b - c) + (c - a)$.
6. $2x - 4y + 2z + (-4x + 2y + 2z)$.
7. $m + n - 2p + (m - 2n + p) + (-2m + n + p)$.
8. $\frac{m}{n} - \frac{a}{b} + \left(2\frac{a}{b} - \frac{m}{n}\right)$.
9. $m + 2n + (m - 2n) + (p + 2q) + (p - 2q)$.
10. $x + y - z + (y + z - x) + (z + x - y)$.
11. $\frac{m}{n} - \frac{a}{b} + \left(\frac{a}{b} - \frac{x}{y}\right) + \left(\frac{x}{y} - \frac{m}{n}\right)$.
12. $ax - by + (2by - 2ax) + (3by - 3ax)$.
13. $2mn - 5xy + (5mn - 2xy)$.

33. *Minus Sign before Parentheses.* If the parentheses are preceded by the sign $-$ we remove them and change the sign of each enclosed term (§ 31).

NOTE. Remember that if a term has no sign before it, then it is positive, and must by this rule be changed to negative.

EXERCISES.

1. $a - 3b - (2b - 5a) + (m + 3a - b) - (a - m)$.

Solution. Changing the signs by the rule, where the parentheses are preceded by the sign $-$, the result is

$$a - 3b - 2b + 5a + m + 3a - b - a + m.$$

The coefficients of a are $+1 + 5 + 3 - 1 = +8$.
" " " b " $-3 - 2 - 1 = -6$.
" " " m " $+1 + 1 = +2$.
Therefore Ans. $= 8a - 6b + 2m$.

2. $x - 3y - (2y - 5x) + (z + 3x - y) - (x - z)$.
3. $6m + 9h - (m - h) + (h - m) - (2m + 2h)$.
4. $a - (-a + b) - (-a + c) - (-a + d)$.
5. $x - y - (y - x) - x + 2y$.
6. $a + b - 2c - (c - 2b + a) - (b - 2a + c)$.
7. $2z - x - y - (z - 2y + x) - (y - 2x + z)$.
8. $ax - (3ax + 2by) + 9ax$.
9. $m - n - (a - b) - (c + d) - (e + g)$.
10. $2am + (3am - n) - (n - 3am) + 2n$.
11. $7h + 7k - (7h - 7k) + (7k - 7h)$.
12. $13px - qy - (-qy - rz) - (rz + 2px)$.
13. $2ap - 3bq - (3ap + 3bq) + (9ap - c - d)$.
14. $a - (x + y - z + u - v + w)$.
15. $x - (x - y + u - w)$.

SECTION III. MULTIPLICATION.

To Multiply a Monomial by a Number.

34. RULE. *Multiply the coefficient, and to the product affix the algebraic symbols of the multiplicand.*

EXAMPLE 1. Multiply $3ax$ by 7.

$$\begin{array}{r} 3ax \\ 7 \\ \hline \text{Ans. } 21ax \end{array}$$

Ex. 2. Multiply ab by 5.
Here the coefficient of ab is 1 and the product is $5ab$.

EXERCISES.

Multiply:

1. $7aby$ by 8.
2. $5mx$ by 6.
3. $9pqr$ by 7.
4. $22b(a + x)$ by 2. Ans. $44b(a + x)$.
5. $3(c + y)m$ by 9.
6. $2(a + b)$ by 3.
7. $a + b$ by 7. Ans. $7(a + b)$.
8. $x - y$ by 8.
9. $2(b - c)$ by 12.

To Multiply one Monomial by Another.

35. RULE. *Multiply the coefficients, and affix to the product all the symbols both of the multiplier and multiplicand.*

EXAMPLE. Multiply $12amx$ by $7acy$.

$$\begin{array}{ll} \text{Multiplicand,} & 12amx \\ \text{Multiplier,} & \underline{7acy} \\ \text{Product,} & 84aacmxy = 84a^2cmxy \end{array}$$

Solution. $7 \times 12 = 84$ is the product of the coefficients, to which we affix the symbols. The symbol a being taken twice as a factor, we write a^2.

EXERCISES.

Multiply:

1. $3ab$ by $12mx$.
2. $5amy$ by $11ab$. Ans. $55a^2bmy$.
3. $7a$ by $6amx$.
4. $6(m+n)$ by $3m$. Ans. $18m(m+n)$.
5. $8(x+y)$ by $9a(x+y)$. Ans. $72a(x+y)^2$.
6. $5(a-b)$ by $3c$.
7. $7(x-z)$ by $3(x+z)$. Ans. $21(x+z)(x-z)$.
8. $19(m-n)$ by $5(p-q)$.
9. $7a(x-y)$ by $6(m-n)$.
10. $8b(c-d)$ by $9q(r-s)$.
11. $2a(p-q)$ by $3b(p-q)$.
12. $5m(m-p)$ by $6m(m-p)$.
13. $7a(b-c)$ by a.
14. ab by ba.
15. xy by $3(x+y)$.
16. $2(a+b)$ by $2(a-b)$.
17. $5mnp$ by $4mnq$.
18. $mn(x+y)$ by $mn(x-y)$.
19. $mp(a+b)$ by $mq(a+b)$.
20. $6ab(x+y)$ by $7ac(x+y)$.

36. *Use of Exponents.* Let us have to multiply

$$a^3 \times a^2.$$

By definition,

$$a^3 = aaa;$$
$$a^2 = aa.$$

Therefore

$$a^3 \times a^2 = aaa \times aa = aaaaa = a^5.$$

Therefore

$$a^3 \times a^2 = a^5.$$

Hence

RULE. *The exponents of like symbols in the factors must be added to form the exponent in the product.*

NOTE. In the following exercises the pupil may advantageously go through the above process in each case, until he fully understands the reason for it.

EXERCISES.

Multiply:

1. $x^3 \times x^2$.
2. $(x + y)^3 \times (x + y)^2$. Ans. $(x + y)^5$.
3. $a^4 \times a^2$.
4. $b^5 \times b^3$.
5. $(a + b)^4 \times (a + b)^2$.
6. $(m + n)^5 \times (m + n)^2$.
7. $2h^4 \times 3h^5$. Ans. $6h^9$.
8. $3m \times 6m^3$ Ans. $18m^4$.

Where there is no exponent, the exponent 1 must be understood.

9. $8a^2b^3 \times 2a^3b$. Ans. $16a^5b^4$.
10. $5m^2n \times 5mn^2$.
11. $4abx \times 5a^2b^2xy$.
12. $5a^3b^4x^2 \times 2a^2bxy$.
13. $8a^2(m + n) \times 9a^3(m + n)^2$. Ans. $72a^5(m + n)^3$.
14. $2(x + y) \times 3(x + y)^2ab$.
15. $3(p + q)^2xy^2 \times 3(p + q)yz$.
16. $6(h + k)^3m^2n \times (h + k)^2mn^2x$.
17. $2bc(b + c) \times b^2c^3(b + c)x$.
18. $5d^2x(d + x)^2 \times 2dx^3(d + x)^4$.
19. $3(m + n)(p + q)^2 \times 2(m + n)^3(p + q)$.
20. $4(a + b)^2(g + h)^2 \times 5(a + b)^4(g + h)$.

37. *Case of more than Two Factors.* When there are three or more factors, multiply two of them, then multiply that product by the third, etc., until all are multiplied.

All the preceding rules may be combined in the following

RULE FOR THE MULTIPLICATION OF MONOMIALS. *Form the product of the numerical coefficients and to it affix all the symbolic factors, giving each the sum of its exponents in the several factors.*

EXERCISES.

Multiply:

1. $ab \times bc \times ca$.

Solution. a, b, c are the factors, and the sum of the exponents of each is 2. Hence

$$ab \times bc \times ca = a^2b^2c^2. \text{ Ans.}$$

2. $xy \times yz \times zx$.
3. $mn \times np \times pm$.
4. $ab \times ac \times ad$. Ans. a^3bcd.
5. $2mx \times 3my \times mz$. Ans. $6m^3xyz$.
6. $2mx \times 3my \times 6mz$.
7. $a \times ab \times abx \times abxy$. Ans. $a^4b^3x^2y$.
8. $2a \times 3ab \times 4abx \times 5abxy$.
9. $2a \times (m + n) \times b$. Ans. $2ab(m + n)$.

REMARK. The pupil will treat quantities in parentheses as mo nomials.

10. $2m \times (x + y) \times a$.
11. $2m \times (x + y) \times 3a$.
12. $3a \times (a + b) \times 6ax$.
13. $4a^2m^2 \times (a + b) \times 2am^2y$. Ans. $8a^3m^4(a + b)y$.
14. $5a^2b^2 \times b^2c^2 \times c^2a^2$.
15. $m^2n^2 \times n^2p^2 \times p^2m^2$.
16. $a^2 \times a^2 \times a$.
17. $ab \times a^3b^2x \times a^4$. Ans. a^8b^3x.
18. $am \times a^2n \times a^3p$.
19. $b^2d \times d^2c \times c^2b$.
20. $b^2 \times 2b^3 \times 3b^4$.
21. $ab^2 \times 2ax^2 \times 2b^2x^2$.
22. $m \times m^2 \times m^3 \times m^4$.
23. $mx \times mx^2 \times mx^3$.
24. $2ax \times 2ay \times 2az \times 2xyz$.
25. $2a^2x^2 \times 3a^2y^2$.

To Multiply a Polynomial by a Monomial.

38. RULE. *Multiply each term of the polynomial by the monomial and take the algebraic sum of the products.*

NUMERICAL EXAMPLE. Multiply $3 + 4 + 5$ by 3.

$$\begin{array}{r} 3 + 4 + 5 \\ 3 \\ \hline 9 + 12 + 15 \end{array} \quad \begin{array}{c} = \\ \\ = \end{array} \quad \begin{array}{r} 12 \\ 3 \\ \hline 36 \end{array}$$

We see by this example that the product of the sum of several numbers, as 3, 4 and 5, is equal to the sum of the pro-

ducts found by multiplying each number separately, because we get the same result, 36, in either case.

ALGEBRAIC EXAMPLE. Multiply $2a + 3bxy + 5ac$ by $7am$.

$$\begin{array}{rl} \text{Multiplicand,} & 2a + 3bxy + 5ac \\ \text{Multiplier,} & 7am \\ \hline \text{Product,} & 14a^2m + 21abmxy + 35a^2mc. \end{array}$$

Operation. Each term is multiplied separately, by the last rule, and the sum taken.

EXERCISES.

Multiply:

1. $a + b$ by x^2.
2. $m + n$ by a.
3. $a + c$ by $2x$.
4. $2a + 3c$ by $3x$.
5. $a + 2b + 3c$ by $2ay$.
6. $2a + 3b + 4c$ by abc.
7. $3ab + 4ac$ by $5abc$.
8. $4ax + 5by + 7cz$ by $3xyz$.
9. $am + bp + cq$ by $2abc$.
10. $6am + 8bn + 9cn$ by $7ax$.
11. $5x+6xy+7xyz$ by $axyz$.
12. $2x + 2y + 9z + u$ by abx.
13. $m+mn+mnp$ by mnp.
14. $p + 2q + 3r$ by $2apq$.
15. Prove the equations

$$\begin{aligned} 9(8 + 7) &= 9 \cdot 8 + 9 \cdot 7. \\ 6(5 - 3) &= 6 \cdot 5 - 6 \cdot 3. \\ 2(1 + 1) &= 2 + 2. \\ 2(1 + 1 + 1) &= 2 + 2 + 2. \\ 7(9 - 5) &= 7 \cdot 9 - 7 \cdot 5. \\ 6(6 - 3) &= 6^2 - 6 \cdot 3. \\ 6(6 - 5) &= 6. \end{aligned}$$

39. If any terms of the polynomial are negative their product by a positive multiplier is negative.

EXERCISES.

Multiply:

1. $a - b$ by m. Ans. $ma - mb$.
2. $m - n$ by a.
3. $a - n$ by b.
4. $2a - 3n$ by c.
5. $-2a + 3n$ by $2b$. Ans. $-4ab + 6bn$.
6. $a - b + c - d$ by x.
7. $a - 2b + 3c - 4d$ by ab.
8. $2a - 3b + 4c$ by abc.
9. $-a + b - 3c$ by axy.

10. $7ab + 2bx - ax$ by $2abx$.
11. $-6am + 7bn - 8cp$ by mnp.
12. $-a^2x + b^2y + c^2z$ by xyz.
13. $2m^2i - 3n^2j + 4p^2k$ by ijk.
14. $7a^2m - 8b^2n - 12c^2p$ by $9mnp$.
15. $12ax^2 - 10by^2 - 7cz^2$ by $8abc$.
16. $-8am^2 + 9bn^2 + 10cp^2$ by $2abc$.
17. $3ab + 2bc - ca$ by abc.
18. $9a^2b - 9ab^2$ by $12xy$.
19. $-12ab - 8bc - 9ca$ by $7abc$.
20. $-xy - 2yz - 3zx$ by $5xyz$.
21. $ax^2 - a^2x$ by axy. Ans. $a^2x^3y - a^3x^2y$.
22. $-2mb + 2m^2b^2$ by $3mbr$.
23. $-3a^2m - 2b^2n$ by $3abmn$.
24. $3ax - 3a^2y$ by $3axy$.

40. *Indicated Multiplications.* The multiplication of a polynomial may be indicated by enclosing it in parentheses and affixing or prefixing the multiplier.

EXERCISES.

Execute the following indicated multiplications:

1. $3(2a - 5bx - c)$. Ans. $6a - 15bx - 3c$.
2. $2(a - 2b + 3cy)$.
3. $4(mx - 2ny + 3pz)$.
4. $4a(ab + ac - ad)$. Ans. $4a^2b + 4a^2c - 4a^2d$.
5. $2m(m^2 + mn^2 + n^3)$.
6. $3b^2(b^2x^2 - c^2y^2)$. Ans. $3b^4x^2 - 3b^2c^2y^2$.
7. $c^2(c^2x^2 - cy^2)$.
8. $c^2y^2(c^2 + y^2)$.
9. $2a^2b^2(3ax - 2by)$.
10. $2a(a + b)ab$. Ans. $2a^3b + 2a^2b^2$.

We first multiply the two factors $2a$ and ab which are without the parentheses, making $2a^2b$. Then we multiply each term within the parentheses by this product.

11. $2m(m + n)mn$.
12. $3h(h - k)hk$.
13. $2m(m - n)mn$.
14. $4a^2(a - b^2)b^2$.
15. $2d^2(x - y)xy$.
16. $7h^2(2m - 3n)k$.
17. $8b(b^2 + h)h^2$.
18. $3a^2b^3(2a - 3h)h$.
19. $2mn^2(m + 2n)3m$.
20. $2ab(a^2 - b^2)2a^2b$.

41. *Negative Multipliers.*

RULE. *When the multiplier is negative, change the signs of all the terms of the product.*

The reason for this rule will be given in Course II.

EXERCISES.

1. $-3(a-2b+3c-4d)$. Ans. $-3a+6b-9c+12d$.
2. $-2(m-2n+3p-4q)$.
3. $-a(x+y-z)$.
4. $-a^2(ax-by+cz)$.
5. $-2ab(-3ax+2by)$. Ans. $6a^2bx-4ab^2y$.
6. $-2ab(a^2-b^2)a$. Ans. $-2a^4b+2a^2b^3$.
7. $-2mn(m^2-n^2)n$.
8. $-3mx(x^2-y^2)xy$.
9. $-8p(x-y)xy$.
10. $-2a(ax-a^2y+a^3z)$.
11. $-m(x-y-z)m^2$.
12. $-3h^3k^2(h-2k^2)h$.
13. $-4hm^2(a-h-2m)2am$.
14. $-4a^2b^2c^2(a^2-b^2+c^2)$.

42. *Combination of Multiplication and Addition.*

EXERCISES.

Execute the following indicated multiplications, and simplify the results by addition:

1. $2a(3x-4y)-a(x+2y)-3a(-x-3y)$.

Work:
$$\begin{aligned} 2a(3x-4y) &= 6ax-8ay \\ -a(x+2y) &= -ax-2ay \\ -3a(-x-3y) &= \underline{3ax+9ay} \\ \text{Sum} &= 8ax-ay \qquad \text{Ans.} \end{aligned}$$

2. $3(a+b)-2(b+c)-(c-a)$.
3. $2a(a+b)+2b(b+a)$. Ans. $2a^2+4ab+2b^2$.
4. $3h(h+m)+3m(m+h)$.
5. $2a(a+b)-2b(b+a)$.
6. $-m(a+b-c)-2a(m-b)$.
7. $a(b-c)-b(c-a)-c(a+b)$.
8. $x(y-z)+y(z-x)+z(x-y)$.
9. $m^2(a^2-b)-a^2(m^2-b)-2b(m^2+a^2)$.
10. $-3(3x-2y)+2x-5y+4(8x-2y)$.
11. $2x(x^2+2x)-3x^2(x-2)-4(6-x^2)$.
12. $2a^2(x^2-a^2)-2x^2(a^2-x^2)$.
13. $2x(x-y)+2y(x-y)$.

14. $3h^2 + 3h(2h - 7) - 2h(3h^2 - 7h - 2)$.
15. $4n^2 - 5n(n - 3) - n(n - 2)$.
16. $2x(x - y)y + 2y(y - x)x$.
17. $2x(a - b) - 2a(x - b) + 2bx$.

SECTION IV. DIVISION.

43. *Def.* The **quotient** is that quantity which multiplied by the divisor will produce the dividend.

The dividend is said to be **exactly divisible** by the divisor when it contains the divisor as a factor.

44. FIRST PRINCIPLE OF DIVISION. *A product may be divided by any of its factors by simply removing them.*

EXAMPLE 1. $3 . 4$, which is 12, may be divided by 4 by removing the 4 leaving 3 as the quotient.

Ex. 2. The product abc may be divided by b by removing b leaving ac as a quotient.

REMARK. When *all* the factors of the dividend are removed the quotient is not 0 but 1.

Ex. 3. $2 . 3$, which is 6, divided by $2 . 3$, which is 6, is 1; ab divided by ab is 1.

Hence, to divide one expression by another, which is an exact divisor of it:

RULE. *Remove from the dividend those factors whose product forms the divisor. The product of the remaining factors will be the quotient. If all the factors are removed the quotient is* 1.

EXERCISES.

1. Divide $3 . 4 . 7$ by $3 . 4$.
2. Divide $6 . 8 . 9$ by 8. Ans. $6 . 9 = 54$.
3. Divide ab by a.
4. Divide $4amn$ by $2m$. Ans. $2an$.
5. Divide $12mxy$ by $4y$.
6. Divide $14bpr$ by $7br$.
7. Divide $7(a + b)$ by $a + b$. Ans. 7.
8. Divide $5m(m + n)$ by $5m$.
9. Divide $6p(r + s)$ by $3(r + s)$.
10. Divide $8u(h + g)$ by $4u(h + g)$.
11. Divide $12d(x + y)f$ by $4f$.

12. Divide $a(x+y)(m+n)$ by $a(m+n)$.
13. Divide $15(f+g)2x(h+g)$ by $5x$.
14. Divide $16(r+s)4mn(u+v)$ by $8m(u+v)$.

45. *Rule of Exponents.* Let us have to divide a^5 by a^3. We have

$$a^5 = aaaaa.$$
$$a^3 = aaa.$$

Hence by the preceding rule $a^5 \div a^3 = aa = a^2$.

Here 2, the exponent in the quotient, is obtained by subtracting the exponent of the divisor from that of the dividend. Hence

Rule. *The exponent of any symbol in the divisor is to be subtracted from the exponent of the like symbol in the dividend.*

Remark. In applying this rule remember that a quantity without any exponent is to be considered as having the exponent 1.

EXERCISES.

Divide:

1. m^5 by m^2. Ans. m^3.
2. m^3 by m.
3. m^8 by m^2.
4. $(a+b)^2$ by $a+b$.
5. am^2 by am.
6. a^2m^3 by am. Ans. am^2.
7. p^3q^4 by pq^2.
8. $12g^2h^5$ by $4g$. Ans. $3gh^5$.
9. $10g^3h$ by $5g$.
10. $9km^2n^3$ by $3nmk$. Ans. $3mn^2$.
11. $8hk^2m^4$ by $2mk$.
12. $6h^2(a+b)^2$ by $6h(a+b)$.
13. $7r^3(x+y)^2$ by $7(x+y)^2r$.
14. $14u^3v(u+v)$ by $2u$.
15. $15(a+b)^3(x+y)^2$ by $5(a+b)(x+y)$.
16. $16ab^2c^3(x+y)^4$ by $4(x+y)abc$.
17. $3a^2m^2(p-q)^3$ by $3a^2m^2(p-q)^2$.
18. $5pq^3(m-n)$ by $pq^3(m-n)$.
19. pa^2xy^3z by $axyz$.
20. $12(a-b)^3 3q^3(r-s)^2$ by $9(a-b)(r-s)q$.
21. $9(a-b)(r-s)$ by $9(r-s)(a-b)$.
22. x^m by x^n. Ans. x^{m-n}
23. y^p by y^q.
24. $(x+h)^m r^h$ by $r^s(x+h)^n$.
25. $a^n x^{2n}$ by $a^n x^n$.
26. $(m+n)^{2n}x^n$ by $(m+n)^n x^n$.
27. $(a-b)^{2n}$ by $(a-b)^{n+1}$.

To Divide a Polynomial.

46. Rule. *Divide each term of the polynomial separately. The sum of the separate quotients will be the quotient required.*

Example. Divide $m^2a + ma^2$ by m.

We first divide m^2a, leaving the quotient ma. Then we divide ma^2, leaving the quotient a^2.

Hence the answer is

$$m^2a + ma^2 \div m = ma + a^2.$$

EXERCISES.

1. $m^2x + mx^2 \div m =$ what?
2. $a^2y + ay^2 \div a$.
3. $9b^2y^3 + 12by^2 \div 3by$.
4. $8p^2x + 4p^3y + 16p^4z \div 4p^2$.
5. $9abx^2 + 12ab^2x + 15a^2bx \div 3abx$.
6. $5h^2(m + n) + 10y^2(m + n) \div m + n$.
7. $(x + y)(x - y) - (x + y)^2 \div x + y$.
8. $(a + b)(a - b)^2 + (a + b)^2(a - b) \div (a + b)(a - b)$.
9. $3x - 6x^2 + 3x^3 \div 3x$.
10. $6m^2(x + y) - 12m^2(x - y) \div 6m$.

47. *Sign of the Quotient.* When the divisor is positive, the quotients of negative dividends must be negative.

Example. If we have to divide $12 - 8$, that is 4, by 2, the quotient from the separate terms will be $6 - 4$. It is evident that the true answer is $6 - 4$, and not $6 + 4$. Hence

$$12 - 8 \div 2 = 6 - 4 = 2.$$

This principle may be expressed in the following form:

Like signs give $+$; *unlike signs give* $-$.

EXERCISES.

1. $9a^2x^2 - 12ax^2 \div 3ax$. Ans. $3ax^2 - 4x$.
2. $8m^2x - 4m^3y - 16m^4z \div 4m^2$.
3. $9pqx^2 - 12pq^2x + 15pq^3x \div 3pqx$.
4. $r^2(m - n)^3 - 5r^3(m - n)^2 \div r(m - n)^2$.
5. $b^2(g - h)^3 - 2b(g - h)^2 \div b(g - h)^2$.
6. $bx + cx^2 - dx^3 - gx^4 + hx^5 \div x$.
7. $5ax - 10a^2y + 5a^3z \div 5a$.
8. $3mu - 6m^2w - 9mw^2 \div 3m$.

9. $5m^2(x-y) - 10m(x-y)^2 \div 5m(x-y)$.
10. $2(m+x)y - 6y^2(m+x)^2 \div 2y$.
11. $3x^2(x+h) - 9x^4(x+h)^3 \div 3(x+h)x$.

Factors and Multiples.

48. *Def.* A **prime expression** is one which has no factors except itself and unity.

Def. A **composite** expression is one which can be expressed as a product of two or more factors.

EXAMPLES. The number 7 is prime because no two numbers multiplied together will make 7.

21 is composite because it is equal to 7×3.

$a + x$ is prime.

$a^2 + ax$ is composite because it is equal to

$$(a+x) \times a$$

which is a product.

49. *Def.* The **degree** of a monomial is the number of literal factors which it contains.

EXAMPLE. The monomial $3ax^3$ is of the fourth degree because it contains the literal factors a, x, x, x.

EXERCISES.

What is the degree of these expressions?

1. bx. 2. bx^2. 3. $2bx^3$.
4. $3c(a+x)$. 5. $7c(a+x)^2$. 6. m^2n^2.

50. *Def.* To **factor** an algebraic expression means to express it as a product of several factors.

PROBLEM. To factor a number or algebraic expression.

RULE. *Find by trial what prime number or expression will divide it; then find what number or prime expression will divide the quotient. Continue the process until a prime quotient is reached. The divisors and last quotient will be the factors required.*

EXAMPLE. Factor the number 420.

$$420 \div 2 = 210;$$
$$210 \div 2 = 105;$$
$$105 \div 3 = 35;$$
$$35 \div 5 = 7.$$

Hence

$420 = 2^2 . 3 . 5 . 7$, which are the factors required.

EXERCISES.

Express the following numbers as products of prime factors:

1. 36. Ans. $2^2 . 3^2$.	2. 12.
3. 28.	4. 81. Ans. 3^4.
5. 256.	6. 324.
7. 72.	8. 140.
9. 56.	10. 82.
11. 100.	12. 52.

51. *Factoring Algebraic Expressions.*

EXAMPLE 1. To factor $b^3 + by$.

We see that b will divide both terms. Dividing by b the quotient is $b + y$ which is prime; therefore

$$b^2 + by = b(b + y).$$

Ex. 2. $a^3 - a^2y = a^2(a - y)$.

EXERCISES.

Factor:

1. $m^2 + mn$.
2. $bx + by$.
3. $bx + by + bz$. Ans. $b(x + y + z)$.
4. $cx - 2cy + 3cz$. Ans. $c(x - 2y + 3z)$.
5. $hp - 3hg + 5hr$.
6. $hp - ahg + bhr$.
7. $2h^2p - 2ah^2g + 4bh^2r$.
8. $n + n^3 + n^5$.
9. $n - 3n^2 + 5n^3$.
10. $4bx^2 - 8b^2x + 12bx$.
11. $3hy + 6h^2yz + 12h^3xyz$.
12. $4r^3 - 8ar^2 - 12r^2$.
13. $4mn^3 + 8m^2n^2 + 4m^3n$.
14. $(a + b)x + (a + b)y$. Ans. $(a + b)(x + y)$.
15. $(m + n)x + (m + n)y$.
16. $(g + h)u - (g + h)v$.
17. $a(x - y) + b(x - y)$.
18. $a(x + y) - b(x + y)$.

Highest Common Divisor.

52. *Def.* A **common divisor** of several quantities is any quantity which will divide them all without a remainder.

Def. The **highest common divisor** of several quantities is their common divisor of highest degree.

Def. When two or more quantities have no common divisor but unity, they are said to be **prime to each other.**

Notation. The highest common divisor is written, for shortness, H. C. D.

53. PROBLEM. To find the H.C.D. of several quantities.

RULE. *Factor each of the quantities. The continued product of the factors common to all, each with its lowest exponent, is their H. C. D.*

EXAMPLE. Find the H. C. D. of 24, 36 and 48.

$$24 = 2^3 . 3; \qquad 36 = 2^2 . 3^2; \qquad 48 = 2^4 . 3.$$

The common prime factors are 2 and 3; the highest exponent of 2 is 2; $\therefore$ $2^2 . 3 = 12$ is the H. C. D.

EXERCISES.

Find the H. C. D. of the following:

1. $54;\ 90;\ 144.$
2. $14;\ 56;\ 63;\ 84.$
3. $72;\ 108;\ 132.$
4. $ax;\ bx;\ 3cx^2.$ Ans. x.
5. $bm;\ bn;\ bk.$
6. $2mn^2x;\ 6mn^3x^2.$
7. $3x^2yz;\ 9x^2y^3z;\ 15x^3y.$ Ans. $3x^2y$.
8. $12m^2n^4;\ 20mn^3;\ 24m^3n^2.$
9. $b(x - h);\ c(x - h).$ Ans. $x - h$.
10. $m^3(m + n);\ mn(m + n).$
11. $3m(g - h);\ 6m^2.$
12. $6m(g - h);\ 12(g - h).$

Lowest Common Multiple.

54. *Def.* The **multiples** of any quantity are all quantities which contain it as a factor.

EXAMPLE. The multiples of 3 are 3, 6, 9, 12, etc., and $3a$, $6b$, $15c$, etc., and in general all quantities which contain 3 as a factor.

Def. A **common multiple** of several quantities is any expression which contains all the quantities as factors.

EXAMPLE 1. 24 is a common multiple of 3, 4, 6, 8, because 24 contains 3 as a factor, 4 as a factor, 6 as a factor and 8 as a factor.

Ex. 2. $ab^2(x + y)$ is a common multiple of a, b, ab, ab^2, $x + y$, etc., because it contains each of them as a factor.

Def. The **lowest common multiple** of several quantities is their common multiple of lowest degree. The lowest common multiple is written, for shortness, L. C. M.

55. PROBLEM. To find the lowest common multiple.

RULE. *Factor each of the quantities. The product of the different factors, each affected with the highest exponent it has in any one of the quantities, is the L. C. M.*

EXAMPLE. Find the L. C. M. of $2xy^2$, $3yz$ and $6x^2z$.

The *different* factors are 2, 3, x, y, z.

The highest exponents, 1, 1, 2, 2, 1.

Lowest common multiple, $2 \cdot 3 \cdot x^2y^2z = 6x^2y^2z$.

Quotients, $3xz$, $2x^2y$, y^2.

EXERCISES.

Find the L. C. M. of the following expressions and the quotients formed by dividing the multiple by the several quantities:

1. $4a^2b^3c$; $13a^3b^2c$; $8abc^2$.
2. $2mn^2$; $3np$; $6m^2p$.
3. qr; rs; st; qt.
4. $4mn$; $8mp$; $4np$.
5. $6gh^2$; $6g^2h$; $3g^2h^2$; $3ghk$.
6. mn^2; nr^2; rs; sm^2.
7. $2u^2vw$; $4uv^3w$; $6u^2v^3w$; $6u$.
8. $3b(m+n)$; $6b^3(m+n)$; $3(m+n)^2$.

   Ans. L. C. M. $= 6b^3(m+n)^2$.

   Quotients $= 2b^2(m+n)$, $m+n$ and $2b^3$.
9. $c^2(m+n)$; $c^2(m+n)^2$; $c(m+n)^3$.
10. $pq^2(g-h)$; $p^2q(g-h)$; $p^3q^2(g-h)$.
11. x; xy; xyz; $xyz(x+y)$.
12. $12h$; $4k$; $3h^2k$.
13. $(a+b)(a-b)$; $(a+b)^2$.
14. $(a-1)^2$; $3b^3(a-1)$; $6b^4(a-1)$.
15. $7(x-y)^2$; $7(x+y)(x-y)$; $14(x+y)^2$.
16. $24(a-b)^2x$; $36(a-b)x^3$.
17. m; m^2n; m^3n^2p; $m^4n^3p^2(m+p)$.
18. ax; $2a^2x$; $3a^3x$; $4a^4x$.
19. $a(x-y)$; $a(x+y)$; $a(x-y)(x+y)$.
20. $abc(p-q)$; $abcd$; $ab(p-q)$.
21. $(m+n)(p-q)$; $(p-q)(m-n)$; $(m-n)(m+n)$.

MISCELLANEOUS EXERCISES.

1. In the division of an estate A got y dollars, B got c dollars more than A, and C got 3 times as much as A and B together. How much did they all get?

2. A pedestrian on the first day walked h hours at the rate of x miles an hour; the second day he walked the same length of time, but one mile an hour faster. How far did he go?

3. If the distance of the earth from the sun is x diameters of the earth, the distance of Jupiter 5 times that of the earth, the distance of Saturn twice that of Jupiter, and the distance of Uranus twice that of Saturn, what is the sum of all these distances?

4. A charitable association working six consecutive days collected x dollars on the first day, and on each of the remaining five days it collected a dollars more than on the day preceding. It then divided the amount equally among $2x$ white and $5a$ colored families. How much did each family get?

5. A library contains $3x$ books of history, $2y$ books less of biography than of history, as much of poetry as of biography and history together, and $3y$ fewer novels than books of poetry. The books were divided equally among 9 alcoves. How many were in each alcove?

6. In winding up a company each stockholder got x dollars, each ordinary creditor got $(h+x)$ dollars more than each stockholder, and each preferred creditor got $2x$ dollars more than each ordinary creditor. There were in all a stockholders, b creditors and c preferred creditors. What was the total amount divided?

7. A railway train having to make a journey of 600 miles ran x hours at the rate of 30 miles an hour and y hours at the rate of 45 miles an hour; it then became disabled and had to make the remaining distance at the rate of 15 miles an hour. How long did it require to make the entire journey?

Method of Solution. We must subtract from the total distance the distances it ran during the x hours and the y hours. The remainder is the distance it had to run at the rate of 15 miles an hour. Dividing this remainder by 15 will give the time required for the last stage of the journey. Adding the times of the first two stages, which are given, we shall have the whole time required, which we shall find to be $(40-x-2y)$ hours.

MEMORANDA FOR REVIEW.

Define and Explain: Use of Parentheses; Value of Symbol; Power; Exponent; Degree; Expression; Coefficient; Term; Monomial; Binomial; Trinomial; Factor; Equation.

Addition and Subtraction.

Define: Like Terms; Algebraic Addition and Subtraction; Numerical Addition and Subtraction; Minuend; Subtrahend; Remainder.

Addition.
- When the terms are unlike.
- Rule for like terms.
- Case of negative terms.
- General rule for positive and negative terms.

Subtraction. Rule; Reason for the rule.

Clearing of Parentheses.
- Sign + before parentheses; Rule.
- Sign − before parentheses; Rule; Reason.

Multiplication.

Define: Multiplier; Multiplicand; Product.

Product of
- A monomial by a number; Rule.
- One monomial by another; Rule.
- Like symbols, using exponents; Rule.
- More than two factors; Method; Rule.
- A polynomial by a monomial; Rule.
- A negative term by a positive multiplier; Principle.
- Any term by a negative multiplier; Rule.

Division.

Define: Dividend; Divisor; Quotient; Exactly Divisible; Prime; Composite; Common Divisor; Highest Common Divisor; Multiple; Common Multiple; Lowest Common Multiple.

Rules and Principles.
- First principle of division; Rule deduced.
- Rule of exponents.
- Division of polynomials; Rule.
- Rule of signs.
- Rule for factoring.

Divisors and Multiples.
- Highest Common Divisor; Rule.
- Lowest Common Multiple; Rule.

CHAPTER III.

ALGEBRAIC FRACTIONS.

56. *Def.* A fraction is the expression of an indicated division formed by writing the divisor under the dividend with a line between them.

EXAMPLE. The quotient of $p \div q$ is the fraction $\frac{p}{q}$.

The **numerator** of the fraction is the *dividend.*

The **denominator** is the *divisor.*

Numerator and Denominator are called **terms** of the fraction.

SECTION I. MULTIPLICATION AND DIVISION OF FRACTIONS.

57. THEOREM. *Multiplying the numerator multiplies the fraction.*

Reason. If we call a fraction $\frac{m}{n}$, it will be the same as m $n$ths. If we multiply it by any factor, as 3, the result will be $3m$ $n$ths, that is $\frac{3m}{n}$.

EXERCISES.

Multiply:

1. $\frac{m}{n}$ by a.
2. $\frac{m}{n}$ by m.
3. $\frac{a^2b}{c^2}$ by ab^2.
4. $\frac{mn}{g}$ by nm.
5. $\frac{1}{pq}$ by p^2.
6. $\frac{1-a^2}{x}$ by bc.
7. $\frac{a(m+n)}{m-n}$ by $7y$.
8. $\frac{1}{a^3}$ by a^2.
9. $\frac{9}{x-y}$ by 9.
10. $\frac{x-y}{x+y}$ by $4xy$.

11. $\dfrac{a^2x^2 - b^2y^2}{ax - by}$ by $9ab$.

12. $\dfrac{3 - m^3}{m^3 + 3}$ by $-3m$. Ans. $\dfrac{3m^4 - 9m}{m^3 + 3}$.

13. $-\dfrac{-a}{b}$ by -1.

14. $\dfrac{a - (b - c)}{a + b - c}$ by $-abc$.

Perform the indicated multiplications:

15. $3a\left(\dfrac{a}{b} - \dfrac{am}{c}\right)$. Ans. $\dfrac{3a^2}{b} - \dfrac{3a^2m}{c}$.

16. $x\left(\dfrac{x}{y} - \dfrac{y}{x} - 1\right)$.

17. $4a^2b\left(\dfrac{m}{n^2} - \dfrac{n^2}{m}\right)$.

18. $3a^2\left(\dfrac{x}{y} + \dfrac{y}{z} + \dfrac{z}{x}\right)b^2c^2$.

19. $-5xy\left(\dfrac{1 - a^2}{1 + a^2} - \dfrac{1 + a^2}{1 - a^2}\right)$.

20. $3m\left(\dfrac{a - b}{c^2} + \dfrac{b - c}{a^2} + \dfrac{c - a}{b^2}\right)n^2xy$.

21. $(m - n)\left(\dfrac{a}{b} + \dfrac{c}{d}\right)$. Ans. $\dfrac{am - an}{b} + \dfrac{cm - cn}{d}$.

22. $(4 + 5)\left(\dfrac{1}{m^2} + \dfrac{2}{n^2} + \dfrac{3}{p^2}\right)$.

23. $-ab\left(\dfrac{a - b}{q^2} - \dfrac{a - b}{q}\right)$.

58. THEOREM. *Dividing the numerator divides the fraction.*

The reason is nearly the same as in the case of multiplication.

EXERCISES.

Divide:

1. $\dfrac{2ax^3}{mn}$ by ax. Ans. $\dfrac{2x^2}{mn}$.

2. $\dfrac{4x^2y^3}{a - b}$ by $2x^2y^2$.

3. $\dfrac{27mn^2}{28q}$ by $9n^2m$.

4. $\dfrac{a(a-b)^3}{b(a+b)}$ by $a-b$.

5. $\dfrac{a(a-b)^3}{b(a+b)}$ by $-(b-a)$.

NOTE. First free the divisor from the parentheses. Comp. § 33, which shows that $-(b-a)$ is the same as $-b+a$ or $a-b$.

6. $\dfrac{7a^2b^3c^4}{13(x-y)}$ by $7cb^2a^2$.

7. $\dfrac{3 \cdot 4 \cdot 5(a+b)(c+d)}{17xy}$ by $5 \cdot 4(c+d)$.

8. $\dfrac{a^2b^3my^2}{k^3}$ by $-amby^2$.

9. $\dfrac{(x^2-y^2)^2}{(x+y)^2}$ by x^2-y^2.

10. $\dfrac{3a^2}{b}-\dfrac{3a^2m}{c}$ by $3a$.

11. $\dfrac{a(m-n)}{b}-\dfrac{c(m-n)}{d}$ by $m-n$.

12. $\dfrac{3x^2y}{q}+\dfrac{15zx^3}{p}$ by $3x^2$.

13. $\dfrac{(a+b)(a-b)}{m}-\dfrac{(a-b)(a+b)}{n}$ by $-(b-a)$.

14. $\dfrac{a^3b^2c}{f}+\dfrac{b^3c^2a}{g}+\dfrac{c^3a^2b}{h}$ by abc.

15. $\dfrac{(1+a)^2}{mn}-\dfrac{(1+a)(1-a)}{nr}$ by $-(1+a)$.

16. $\dfrac{-(1-a^2)}{m}+\dfrac{(a+1)(a^2-1)}{n}$ by a^2-1.

17. $\dfrac{ab(x^2+y^2+z^2)}{c}-\dfrac{b(y^2+z^2+x^2)c}{a}+\dfrac{(z^2+x^2+y^2)ca}{b}$
by $x^2+y^2+z^2$.

18. $\dfrac{14a^3b^5m^2}{1-a}-\dfrac{70m^3b^5a^8}{1+a}$ by $7a^2b^4m^2$.

19. $\dfrac{f-g}{k}+\dfrac{-2(g-f)}{m}$ by $f-g$.

20. $\dfrac{k^2}{a}-\dfrac{k^3(1-k)}{b}+\dfrac{k^5(1-k^2)}{c}$ by k^2.

21. $\dfrac{a}{1}-\dfrac{a^3}{2 \cdot 3}+\dfrac{a^5}{2 \cdot 3 \cdot 4 \cdot 5}$ by $-a$.

59. THEOREM. *Multiplying the denominator divides the fraction.*

Reason. Suppose a line first divided into 4 parts. One of the parts will then be $\frac{1}{4}$ of the line and n of them will then be $\frac{n}{4}$ of it.

Now multiply the denominator 4 by 3. This will be dividing the line into $3 \cdot 4 = 12$ parts, so that each fourth will be divided into 3 parts. Therefore $\frac{1}{4} \div 3 = \frac{1}{12}$, and $\frac{n}{4} \div 3 = \frac{n}{12}$.

EXERCISES.

1. $\frac{a^2}{7} \div 3$. Ans. $\frac{a^2}{21}$.

2. $\frac{3a}{5k} \div 5$.

3. $\frac{1}{5k} \div 5k$.

4. $\frac{1}{n} \div 2n$.

5. $\frac{8}{1-a} \div 4(1-a)$.

6. $\frac{q}{1-q} \div q(1+q)$.

7. $\frac{1}{(q+1)} \div -(1-q)$.

8. $\frac{a^2}{(a+4)(a-4)} \div a(a-4)$.

9. $\frac{6^2}{(6+4)(6-4)} \div 6(6-4)$.

10. $\frac{p^2(1+p)}{q^2(1+q)} \div 3p^2q^2$.

11. $\frac{1}{ab} - \frac{1}{bc} \div abc$.

12. $\frac{1}{3 \cdot 4} - \frac{1}{4 \cdot 5} \div 3 \cdot 4 \cdot 5$.

13. $x^2 + 1 + \frac{1}{x^2} \div x^2$. Ans. $1 + \frac{1}{x^2} + \frac{1}{x^4}$.

14. $\frac{2}{k^2} + \frac{k^2}{2} \div 2k^2$.

15. $a - \frac{a^2}{2} + \frac{a^3}{3} - \frac{a^4}{4} \div a^2$.

16. $\frac{x^2}{y} - \frac{y}{x^2} \div axy$.

17. $\frac{1}{a^2x^2} - \frac{b}{a^2x} - \frac{c}{2ax^2} \div ax$.

60. THEOREM. *Dividing the denominator multiplies the fraction.*

Reason. If the fraction is $\frac{1}{12}$, and we divide the denominator 12 by 3 we have $\frac{1}{4}$, which is 3 times as much as $\frac{1}{12}$. So also $\frac{n}{4}$ is 3 times as much as $\frac{n}{12}$, or

$$\frac{n}{12} \times 3 = \frac{n}{4}.$$

EXERCISES.

1. $\frac{a}{6} \times 2.$

2. $\frac{a}{6} \times 3.$

3. $\frac{a}{6} \times 6.$

4. $\frac{a}{24b} \times 12.$

5. $\frac{a-b}{(a+b)^2} \times (a+b).$

6. $\frac{3x^2y^3}{4m^3n^2} \times -2m^2n.$

7. $\left(\frac{a}{2b^2} - \frac{3a^2}{4b^3}\right) \times -2b^2.$

8. $\frac{1-k^2+k^4}{12(m-n)} \times (n-m).$

NOTE. $n - m = -(m-n).$

9. $\frac{1}{2(3-9a^2+18a^4)} \times 6.$

10. $\frac{1}{a-b} \times (b-a).$

11. $\frac{1-x}{(1+x)^2} \times (1+x).$

12. $\left\{\frac{a}{3} - \frac{a^3}{2\,.\,3} + \frac{a^5}{2\,.\,3\,.\,5}\right\} \times 3.$

13. $\left\{1 + \frac{2}{a^2} + \frac{4}{a^4} + \frac{6}{a^8}\right\} \times a^2.$

14. $\left\{1 - \frac{2}{a^3} + \frac{4}{a^4} - \frac{6}{a^8}\right\} \times a^3.$

15. $\left(\frac{1}{a^2b^2c^2} - \frac{2}{a^2b^2d^2}\right) \times a^2b^2.$

16. $\left\{\frac{m}{n} - \frac{m+n}{m^2}\right\} \times m.$

17. $\left(\frac{p}{4a^2xy} - \frac{q}{8ax^2y}\right) \times 4xy.$

18. $\left\{\frac{1}{n^2} - \frac{1}{n^3} + \frac{1}{n^4}\right\} \times n.$

19. $\left\{\frac{a}{(x-y)^2} + \frac{b}{(x-y)^3} + \frac{c}{(x-y)^4}\right\} \times (x-y).$

20. $\frac{m+n}{a^3x+a^3y} \times (x+y).$

21. $\frac{1}{a^2b+a^3} \times a+b.$

22. $\frac{p+q}{a^2m+m} \times (a^2+1).$

23. $\frac{m-n}{ax-bx} \times x.$

61. If we multiply a fraction $\frac{p}{q}$ by its denominator q we have by the preceding rule $\frac{p}{1}$, that is p, as the product. Therefore

A fraction is multiplied by its denominator by simply removing it.

EXAMPLES. $\frac{1}{2} \times 2 = 1$; $\frac{2}{a} \times a = 2$; $\frac{m}{1-k^2} \times (1-k^2) = m$.

62. When the multiplier and the denominator contain common factors, we may multiply by these factors by removing them from the denominator, and then multiply the remaining factors into the numerator.

EXAMPLE. Multiply $\frac{a}{mn}$ by mx.

We multiply by m by removing it from the denominator, and then multiply the numerator by x. Therefore

$$\frac{a}{mn} \times mx = \frac{ax}{n}.$$

EXERCISES.

1. $\frac{a}{b^2c^2} \times ac^2$.
2. $\frac{3qk}{4x^2y} \times 12qkx$.
3. $\frac{(1-x)}{(1+x)^2} \times (1-x)(1+x)$.
4. $\left(\frac{a^2}{b} - \frac{b}{a^2}\right) \times a^2b^2$.
5. $\left(\frac{m}{a^2} + \frac{n}{a} + p\right) \times a^3mnp$.
6. $\frac{3k}{1-a} \times (a-1)(a+1)$.
7. $\frac{3}{7x^2y^2} \times 14x^3y^3$.
8. $\left\{\frac{a}{1-a} + \frac{b}{1-b} + \frac{c}{1-c}\right\} \times (1-a)(1-b)(1-c)$.
9. $\left\{x^2 + 2 + \frac{1}{x^2}\right\} \times ax^2$.
10. $\frac{s}{(s-a)(s-b)(s-c)} \times s(s-a)$.

11. $\dfrac{1}{(1-q)(1-p)} \times -a(q-1)(1-p)$.

12. $\left(\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}-1\right) \times a^2b^2$.

SECTION II. REDUCTION OF FRACTIONS.

63. *Def.* Reduction means changing the form of an expression without changing the value.

64. *Reduction to lowest terms.*

THEOREM. *If both terms of a fraction be multiplied or divided by the same quantity, the value of the fraction will not be altered.*

Reason. By §§ 57, 58, 59 and 60 the multiplications of the two terms have opposite effects, which cancel each other, and so have the divisions. Hence all common *factors* in the two terms may be cancelled.

NOTE. Observe that only *factors* can be cancelled.

65. *Def.* When all the common factors are cancelled the fraction is said to be reduced to its **lowest terms.**

To reduce a fraction to its lowest terms:

RULE. *Factor each term when necessary, and cancel all factors common to both.*

EXAMPLE 1. $$\frac{mnp^2x^2}{rsp^2x^2}=\frac{mn}{rs}.$$

The factors p^2x^2, common to both terms, are cancelled.

Ex. 2. $$\frac{x^3y}{x^2y^3}=\frac{x}{y^2},$$

the factors x^2y being cancelled.

Ex. 3. $$\frac{mx-nx}{my-ny}=\frac{(m-n)x}{(m-n)y}=\frac{x}{y}.$$

EXERCISES.

Reduce to lowest terms:

1. $\dfrac{ab^2c^3}{a^3b^2c}$.

2. $\dfrac{ab+ac^2}{c^2a-ab^2}$.

3. $\dfrac{mn^2xy - n^2x}{n^2x(1-a)}$.

4. $\dfrac{pq^2 - p^2q^3 + p^3q^4}{p^2q^2 + p^4q^4 + p^6q^6}$.

5. $\dfrac{(1-x)^2\,(1+x)^2}{(1-x)\,(x+1)}$.

6. $\dfrac{(a-1)\,(a-2)}{(1-a)\,(2-a)}$.

NOTE. $a - 1 = -(1-a)$.

7. $\dfrac{18a^2b^3c^4}{72a^3b^2cxy^2}$.

8. $\dfrac{amx - amy}{m^2a^2x + ma^3y}$.

9. $\dfrac{rs - r^2s^3}{r^2s^2}$.

10. $\dfrac{f^3x^2 + f^4y^4}{abcdf^3}$.

11. $\dfrac{(1 - a^2 + a^4)xy}{xy + a^4xy}$.

12. $\dfrac{7\,.\,9 + 3}{7\,.\,9 - 2\,.\,3}$.

13. $\dfrac{a(q-k)x^2y}{3a^3(k-q)xy}$.

14. $\dfrac{a^2}{abc} + \dfrac{b^2}{abc} + \dfrac{c^2}{abc}$.

15. $\dfrac{x^3}{xy^2} + \dfrac{y^3}{yz^2} + \dfrac{z^3}{zx^2}$.

16. $\dfrac{3^2}{3\,.\,4\,.\,5} + \dfrac{4^2}{3\,.\,4\,.\,5} + \dfrac{5^2}{3\,.\,4\,.\,5}$.

66. *Reduction to Given Denominator.* An entire quantity may be expressed as a fraction with any required denominator, D, by supposing it to have the denominator 1 and then multiplying both terms by D.

For if a is any entire quantity, we have

$$a = \frac{a}{1} = \frac{aD}{D}.$$

EXAMPLE. If we wish to express the quantity ab as a fraction having xy for its denominator, we write

$$\frac{abxy}{xy}.$$

67. If the quantity is fractional, both terms of the fraction must be multiplied by that factor which will produce the required denominator.

EXAMPLE. To express $\dfrac{a}{b}$ with the denominator nb^3 we multiply both members by $nb^3 \div b = nb^2$. Thus,

$$\frac{a}{b} = \frac{anb^2}{nb^3}.$$

This process is the reverse of reducing to lowest terms.

EXERCISES.

Express the quantity:

1. m with denominator a. Ans. $\frac{am}{a}$.

2. m with denominator m. Ans. $\frac{m^2}{m}$.

3. $a - b$ with denominator x.

4. $\frac{p}{q}$ with denominator qr.

5. $\frac{3}{4}$ with denominator 12.

6. ab with denominator $a + b$.

7. $x - y$ with denominator $x + y$.

8. $\frac{1 - k^2}{a}$ with denominator abc.

9. $\frac{1 - k^2 + k^4}{2xy}$ with denominator $10ax^2y^2(k^2 - 1)$.

10. $\frac{a^n + a^{n-1}}{- 4m}$ with denominator $20m^2$.

11. $\frac{1 - g^2}{g - 1}$ with denominator $1 - g$.

Note. $1 - g = - (g - 1)$.

12. $\frac{a}{bc}$, $\frac{b}{ca}$ and $\frac{c}{ab}$ with denominator abc.

13. $\frac{1}{r}$, $\frac{1}{s}$ and $\frac{1}{t}$ with denominator rst.

14. $\frac{x^2}{a^2}$, $\frac{y^2}{b^2}$ and 1 with denominator a^2b^2.

15. $\frac{s}{s - a}$, $\frac{s}{s - b}$ and $\frac{s}{s - c}$ with den. $(s-a)(s-b)(s-c)$.

16. $\frac{a - b}{- a}$ with denominator $+ a$.

17. $\frac{a}{b}$ and $\frac{x}{y}$ with denominator $abxy$.

68. To reduce fractions to a common denominator:

RULE. *Choose a common multiple of the denominators. Multiply both terms of each fraction by the multiplier necessary to change its denominator to the chosen multiple.*

NOTE 1. Any common multiple of the denominators may be taken as the common denominator, but the L. C. M. is the simplest.

NOTE 2. The required multipliers will be the quotients found by dividing the chosen multiple by the denominator of each separate fraction.

NOTE 3. When the denominators have no common factors, the multiplier for each fraction will be the product of the denominators of all the other fractions.

NOTE 4. An entire quantity must be regarded as having the denominator 1. (§ 66.)

EXAMPLE 1. Reduce to common denominator

$$m, \quad \frac{1}{m}, \quad \frac{p}{mn} \quad \text{and} \quad \frac{m}{np}.$$

The given denominators are 1, m, mn, np.

Their L. C. M. is mnp.

The multipliers are (Note 2) mnp, np, p and m.

Multiplying by these quantities, the fractions become

$$\frac{m^2np}{mnp}, \quad \frac{np}{mnp}, \quad \frac{p^2}{mnp} \quad \text{and} \quad \frac{m^2}{mnp}.$$

Ex. 2. $$\frac{x}{a}, \quad \frac{y}{b}, \quad \frac{z}{c}.$$

By Note 3 the multipliers are bc, ac and ab.

Multiplying by them, the fractions become

$$\frac{bcx}{abc}, \quad \frac{acy}{abc} \quad \text{and} \quad \frac{abz}{abc}.$$

EXERCISES.

Reduce to common denominator:

1. $1, \quad \frac{a}{b}, \quad \frac{b}{a}.$

2. $\frac{1}{a}, \quad \frac{1}{b}, \quad \frac{1}{c}.$

3. $\frac{b}{a}, \quad \frac{c}{b}, \quad \frac{a}{c}.$

4. $\frac{m^2}{a^2}, \quad \frac{m}{a}, \quad m.$

5. $\frac{x}{2y^2}, \quad \frac{3}{4y^3}.$

6. $\frac{2x}{a}, \quad \frac{3y}{1-a}, \quad \frac{4z}{a^2}.$

7. $1, \quad \frac{2a}{3b}, \quad \frac{2 \cdot 3a^2}{3 \cdot 4b^2}.$

8. $1, \quad \frac{1}{m}, \quad \frac{1}{n}, \quad \frac{1}{mn}.$

9. $\frac{x}{3}, \quad \frac{x}{4}, \quad \frac{x}{5}.$

10. $\frac{1}{3x}, \quad \frac{1}{4y}, \quad \frac{1}{5z}.$

11. $\frac{A^2}{a}, \quad \frac{B^2}{b}, \quad \frac{C^2}{c}.$

12. $\frac{3}{-a}, \quad \frac{m}{-ab}, \quad \frac{5n}{b}.$

13. $f, \quad \frac{-f}{g}, \quad \frac{-f^2}{mg}.$

14. $a, \quad \frac{-a^3}{2 \cdot 3}, \quad \frac{a^5}{2 \cdot 3 \cdot 4 \cdot 5}.$

15. $\frac{1}{m-1}, \quad \frac{2}{1-m}, \quad \frac{3}{m}.$

16. $\frac{a}{1-x}, \quad \frac{b}{2x(1-x)}, \quad \frac{c}{3x^2}.$

SECTION III. AGGREGATION AND DISSECTION OF FRACTIONS.

69. *Def.* **Aggregation** is the expression of the algebraic sum of several fractions as a single fraction.

Def. **Dissection** is the separation of a fraction into an algebraic sum of fractions.

70. CASE I. When the fractions to be aggregated have the same denominator.

RULE. 1. *Enclose each numerator between parentheses, or suppose it so enclosed.*

2. *Prefix to each numerator so enclosed the algebraic sign of the fraction.*

3. *Form the algebraic sum of the expressions thus found and write the common denominator under them.*

Reason. 1. By the definition of a fraction, the numerator expresses a number of fractional units.

2. Hence the sum of several fractions with a common denominator is the sum of all the fractional units indicated by the several numerators.

3. If the fractions are preceded by the minus sign, it indicates that the fractional units in its numerator are to be algebraically subtracted. This subtraction is indicated by enclosing the numerator between parentheses and prefixing the minus sign.

4. Hence the algebraic sum of all the fractions is formed in the manner directed in the rule.

EXAMPLE 1. $\frac{a}{d}-\frac{b}{d}+\frac{c}{d}$ means b fractional units to be subtracted from a fractional units, and c fractional units to be added. Hence

$$\frac{a}{d}-\frac{b}{d}+\frac{c}{d}=\frac{a-b+c}{d}.$$

Ex. 2.

$$\frac{c+5x}{D}-\frac{3c-2x}{D}-\frac{4x-8c}{D}=\frac{c+5x-(3c-2x)-(4x-8c)}{D}$$
$$=\frac{c+5x-3c+2x-4x+8c}{D}$$
$$=\frac{6c+3x}{D}.$$

EXERCISES.

Aggregate:

1. $\frac{a-x}{m}+\frac{a+x}{m}$. Ans. $\frac{2a}{m}$.

2. $\frac{a+x}{m}-\frac{a-x}{m}$. Ans. $\frac{a+x-(a-x)}{m}=\frac{2x}{m}$.

3. $\frac{a+2x}{m}-\frac{a-x}{m}$.

4. $\frac{2a-3x}{m}-\frac{a+3x}{m}$.

5. $\frac{a+b}{m-n}-\frac{2a-3b}{m-n}$.

6. $\frac{y}{x-y}-\frac{x}{x-y}$.

7. $\frac{3x}{x+a}+\frac{3a}{x+a}$.

8. $\frac{a-2b+3c}{1-x^2}+\frac{a+2b-3c}{1-x^2}$.

9. $\frac{s-a}{s}+\frac{s-b}{s}+\frac{s-c}{s}$.

10. $\frac{2f-g}{m}-\frac{f-2g}{m}$.

When $a+b+c=2s$, what does the answer to Ex. 9 reduce to?

11. $\frac{1-a}{4xy^2}+\frac{1-b}{4xy^2}+\frac{a+b}{4xy^2}$.

12. $\frac{a^2+2ab+b^2}{mn}-\frac{a^2-2ab+b^2}{mn}$.

13. $\frac{m(1-a)}{k^2}+\frac{m(a-1)}{k^2}$.

14. $\frac{2(1-k^2)}{2.3.4}-\frac{1-k^4}{2.3.4}$.

15. $\frac{2a+b}{D}-\frac{2a-b}{D}+\frac{5a-2b}{D}-\frac{a+2b}{D}$.

71. CASE II. If an entire quantity and a fraction are to be aggregated, we reduce the entire quantity to a fraction having the same denominator as the fraction.

EXAMPLE. Aggregate $a + \frac{m}{n}$.

Reducing by § 66, we have $a = \frac{an}{n}$. So the sum is

$$\frac{an}{n} + \frac{m}{n} = \frac{an+m}{n}.$$

We now see that the result is obtained by the following rule:

Multiply the entire quantity by the denominator, and add the product with the numerators.

EXERCISES.

Aggregate:

1. $1 - \frac{a}{1-x}$. Ans. $\frac{1-x-a}{1-x}$.

2. $1 + \frac{x}{1-x}$.

3. $1 - \frac{x}{1+x}$.

4. $a + \frac{a}{m-1}$.

5. $x + \frac{x-y}{x+y}$.

6. $f + \frac{1}{f}$.

7. $x + y - 1 - \frac{x-y+1}{5}$.

8. $1 - \frac{1}{1+a}$.

9. $4mn - \frac{a-b}{4^2m^2n^2}$.

10. $x + \frac{1-x}{x+2}$.

11. $A - \frac{3}{4+5a^2}$.

12. $m - k + \frac{1}{m}$.

13. $4Aa - \frac{A}{a+A}$.

14. $x + \frac{1-x^2}{x^3}$.

15. $\varepsilon + \frac{1}{a^{\varepsilon}}$.

16. $x + a + \frac{1}{a^x}$.

17. $1 + x^n - \frac{1-x^m}{x^n}$.

18. $\frac{a^2+2ab+b^2}{3amx^2} - \frac{4ab}{3amx^2} + \frac{a^2-2ab+b^2}{3amx^2}$.

72. Case III. When the fractions have different denominators we reduce them to a common denominator and proceed as in § 70.

EXERCISES.

Aggregate:

1. $\frac{1}{a} + \frac{1}{b}$. Ans. $\frac{a+b}{ab}$.
2. $\frac{1}{b} - \frac{1}{a}$.
3. $\frac{m}{a} - \frac{m}{b}$.
4. $\frac{m}{m-n} - \frac{n}{n}$.
5. $\frac{m}{m+n} + \frac{n}{m}$.
6. $\frac{a}{a-b} - \frac{3b}{a}$.
7. $\frac{a}{2} + \frac{b}{3} + \frac{c}{4}$.
8. $x + \frac{x^2}{c} + \frac{x^3}{c^2}$.
9. $\frac{1}{ab} - \frac{1}{bc} + \frac{1}{ca}$.
10. $\frac{a-b}{c} + \frac{b-c}{a} + \frac{c-a}{b}$.
11. $1 + \frac{a}{b} + \frac{b}{c} + \frac{c}{a}$.
12. $1 + \frac{1}{s} + \frac{1}{s^2} + \frac{1}{s^3}$.
13. $1 - \frac{f^2}{2} + \frac{f^4}{2 \,.\, 3 \,.\, 4}$.
14. $f - \frac{f^3}{2 \,.\, 3} + \frac{f^5}{2 \,.\, 3 \,.\, 4 \,.\, 5}$.
15. $1 - \frac{1}{a^x}$.
16. $\frac{1}{a^x} - \frac{1}{b^y} + 1$.
17. $\frac{1}{m^e} - \frac{1}{m^{2e}}$.
18. $\frac{1}{m^3} - \frac{1}{m^{2 \cdot 3}}$.

73. *Dissection of Fractions.* If the numerator is a polynomial, each of its terms may be divided separately by the denominator, and the several fractions connected by the signs + or −.

The principle is that on which the division of polynomials is founded (§ 46). The general form is

$$\frac{A + B + C + \text{etc.}}{m} = \frac{A}{m} + \frac{B}{m} + \frac{C}{m} + \text{etc.} \qquad (1)$$

The separate fractions may then be reduced to their lowest terms.

Example. Dissect the fraction

$$\frac{12abc + 6ab - 3c}{12abc}.$$

The general form (1) gives for the separate fractions

$$\frac{12abc}{12abc}+\frac{6ab}{12abc}-\frac{3c}{12abc}.$$

Reducing each fraction to its lowest terms the sum becomes

$$1+\frac{1}{2c}-\frac{1}{4ab}.$$

EXERCISES.

1. $\dfrac{a^2-x^2+2ax}{2a^2}$.
2. $\dfrac{x^2-2xy+y^4}{x^2y^2}$.
3. $\dfrac{ab+bc+ca}{abc}$.
4. $\dfrac{am^2-2amn-a^2b^2}{a^2m^2}$.
5. $\dfrac{A+B}{AB}$.
6. $\dfrac{(f-g)+(h-j)}{(f-g)\,(h-j)}$.

NOTE. $f-g$ and $h-j$ are to be considered as single quantities; for example, like A and B in Ex. 5.

7. $\dfrac{a^2y^2+b^2x^2-a^2b^2}{a^2b^2}$.
8. $\dfrac{1+a+a^2+a^3}{a^3}$.
9. $\dfrac{p^2+q^2+r^2}{2pqr}$.
10. $\dfrac{3^2+4^2+5^2}{2\,.\,3\,.\,4\,.\,5}$.
11. $\dfrac{x^2-2px-q}{2pqx}$.
12. $\dfrac{a(1-x)-b(1+x)}{ab}$.
13. $\dfrac{a(1-x)-b(1+x)}{(1-x)\,(1+x)}$.
14. $\dfrac{a^n-b^n}{a^nb^n}$.
15. $\dfrac{a^n+b^n}{a^{2n}b^{2n}}$.
16. $\dfrac{A^3-B^3}{A^{2\cdot 2}B^{2\cdot 2}}$.

SECTION IV. MULTIPLICATION AND DIVISION BY FRACTIONS.

74. *Multiplication by Fractions.*

Def. To multiply a quantity by a fraction means to take the same part of the quantity that the fraction is of unity.

EXAMPLE. To multiply Q by $\frac{m}{n}$ means to take $\frac{m}{n}$ of Q.

To form $\frac{m}{n}$ of Q, we divide Q into n parts, and take m of those parts. Hence:

To multiply by a fraction, we divide the multiplicand by the denominator and multiply the result by the numerator.

EXERCISES.

Multiply:

1. $a+b$ by $\frac{a}{b}$. Ans. $\frac{a^2}{b}+\frac{ab}{b}=\frac{a^2}{b}+a$.

2. $a-b$ by $\frac{b}{a}$.

3. p^2+q^2 by $\frac{p^2}{q^2}$.

4. $m+n-1$ by $\frac{1}{mn}$.

5. $m-n+1$ by $\frac{m}{n}$.

6. a^2b-ab^2 by $-\frac{1}{ab}$.

7. $a+a^2+a^3$ by $\frac{1}{a^2}$.

8. $4xy+7ax^2-11by$ by $\frac{ab}{xy}$.

9. $ak-bk^2$ by $-\frac{ab}{k^2}$.

10. $a^n+a^{2n}+a^{3n}$ by $\frac{1}{a^{2n}}$.

11. $a(f-g)-b^2(f-g)$ by $\frac{a}{b(f-g)}$.

12. $qk-x^2$ by $\frac{-x^2}{qk}$.

13. $(1-a)b-a(1-b)$ by $\frac{ab}{(1-a)(1-b)}$.

14. $1-x-y-z$ by $\frac{m}{xyz}$.

15. $2a^n+3b^n$ by $\frac{2a^n}{3b^n}$.

75. *The Multiplicand a Fraction.* If the multiplicand is also a fraction, suppose $\frac{a}{b}$, and is to be multiplied by $\frac{m}{n}$, then by §§ 57, 59, we multiply by m by multiplying the numerator, a, and divide by n by multiplying the denominator, b.

Hence $$\frac{a}{b}\times\frac{m}{n}=\frac{am}{bn}.$$

That is, *we multiply the numerators for a new numerator, and the denominators for a new denominator.*

The result should then be reduced to lowest terms.

EXERCISES.

Execute the following multiplications:

1. $\frac{a}{b}\times\frac{mb}{pa}$. Ans. $\frac{amb}{apb}=\frac{m}{p}$.

2. $\frac{gx^2}{pq}\times\frac{3p^2x}{gr}$.

3. $\frac{2am^2x}{3hy}\times\frac{2my}{3hx}$.

4. $\frac{a-b}{m}\times\frac{m-n}{a-b}$.

5. $\left(\frac{a}{b}+\frac{b}{a}\right)\frac{a^2}{b^2}$.

6. $\left(\frac{p}{q}-\frac{p^2}{q^2}\right)\frac{q^2}{p^2}$.

7. $\left(\frac{c}{r}+\frac{c^2}{r^2}+\frac{c^3}{r^3}\right)\frac{r}{c}$.

8. $\left(\frac{h}{1-r}-\frac{k}{1+r}\right)\frac{1-r}{1+r}$.

9. $\left(a^2+ac+\frac{a^2c}{b}\right)\frac{b}{a}$.

10. $\left(\frac{9b}{2}-\frac{6b^2}{a}+\frac{15bc}{2a}\right)\frac{2a}{3b}$.

11. $\left(xy^2-2y^3+\frac{y^4}{x}\right)\frac{x}{y^2}$.

12. $\left(a-\frac{a^3}{3}+\frac{a^5}{5}\right)x-\frac{1}{a^3}$.

13. $\left(\frac{1+a}{m^2}-\frac{n^2}{1-a}\right)\frac{mn(1-a)}{1+a}$.

14. $\left(\frac{a^n}{b^n}-\frac{b^n}{a^n}\right)\frac{a^n}{b^n}$.

15. $\left(\frac{ax}{yz}+\frac{ay}{zx}+\frac{az}{xy}\right)\frac{xyz}{a}$.

76. *Division by Fractions.*

Def. To divide a quantity by a fraction means to find that expression which, when multiplied by the fraction, will produce the quantity.

If the dividend is $\frac{a}{b}$ and the divisor is $\frac{m}{n}$, then the quotient must be $\frac{na}{mb}$ because $\frac{na}{mb}\times\frac{m}{n}=\frac{a}{b}$. Hence, to divide by a fraction,

Invert the terms of the divisor and multiply by the fraction thus formed.

EXERCISES.

Execute the following divisions:

1. $\frac{r}{s} \div \frac{r}{s}$.
2. $\frac{2r}{s} \div \frac{r}{2s}$.
3. $a \div \frac{b}{a}$.
4. $m \div \frac{1}{m}$.
5. $1 \div \frac{am}{bn}$.
6. $\frac{1}{a} \div \frac{m}{b}$.
7. $\frac{p^2}{q^2} \div \frac{q^2}{p^2}$.
8. $\frac{p^a}{q^a} \div \frac{q^b}{p^b}$.
9. $\frac{4amx^2}{5bny^2} \div \frac{4am}{5bn}$.
10. $\left(\frac{a}{b} + \frac{b}{c}\right) \div \frac{a}{c}$.
11. $\left(\frac{a}{x^2} - \frac{2a}{y^2}\right) \div \frac{x^2y^2}{a^2}$.
12. $\left(\frac{m}{1-a} - \frac{1+a}{n}\right) \div \frac{1+a}{1-a}$.
13. $\left(\frac{s}{s-a} + \frac{s}{s-b}\right) \div \frac{s^2}{(s-a)(s-b)}$.
14. $\left(1 - \frac{1}{a}\right) \div \frac{1}{1-a}$.
15. $\left(\frac{qr}{s} + \frac{rs}{q} + \frac{r}{sq}\right) \div \frac{1}{qrs}$.
16. $\frac{17a^2b^2xy^n}{19mnz} \div \frac{51m^2n^2x}{19a^4by^n}$.
17. $\frac{13}{a-1} \div \frac{14}{1-a}$.
18. $\frac{a}{9} \div -\frac{1}{9a}$.
19. $\frac{1}{x^a} \div \frac{a}{x^b}$.
20. $\left(\frac{1}{a^x} - \frac{1}{a^{2x}}\right) \div \frac{1}{a^{3x}}$.
21. $1 \div \frac{A}{B}$.

NOTE. The answers to Exercises 5 and 21 show that the quotient of unity divided by any fraction is equal to the fraction inverted.

MISCELLANEOUS EXERCISES.

Express and reduce:

1. $\frac{2}{3}$ of $\frac{2}{5}$ of $7x$.
2. $\frac{1}{3}$ of $\frac{5}{2}$ of $2x$.
3. $\frac{3}{5}$ of $\frac{1}{2}$ of $(5x - a)$.
4. $\frac{1}{5}$ of $\left(\frac{2}{5}x - a\right)$.
5. $\frac{1}{m}$ of $\left(\frac{m}{n}x + na\right)$.
6. $\frac{1}{m}$ of $\frac{m}{n}$ of $\frac{n}{p}$ of x.
7. From a sum of x dollars $\frac{1}{3}$ of the amount and a dollar more were taken. Express $\frac{1}{3}$ the remainder.

8. Increase the quantity x by $\frac{1}{n}$ part of itself, and express the result as a fraction, with x as a factor.

9. Diminish the quantity x by the nth part of itself, and express the result in the same way.

10. From a cistern containing g gallons of water $\frac{1}{3}$ the water was taken one day and $\frac{1}{3}$ of what was left the next day. Express the remainder.

11. A father left b thousand dollars to each of a children. Each of these children had b other children between whom the money was equally divided. How much did each grandchild get?

12. A charitable association collected x dollars from each of a people and y dimes from each of b people. It divided the amount equally among a almoners, and each of these almoners divided his share among b poor. How much did each poor person get?

13. A man having to make a journey of m miles went $\frac{1}{3}$ the distance and a miles more on the first day, and $\frac{1}{2}$ the remaining distance $+ b$ miles on the second day. How far had he still to go?

14. A huckster with t turkeys sold half of them at $\frac{m}{n}$ dollars each, and $\frac{1}{3}$ the remainder at $\frac{m^2}{n^2}$ dollars each, and what were left at $\frac{m^3}{n^3}$ dollars each. How much did he realize?

MEMORANDA FOR REVIEW.

The Two Conceptions of a Fraction.

1. As parts of a unit.
2. As the quotient of an indicated division.

Explain significance of numerator and denominator in each mode of conception.

Multiplication and Division of Fractions.

Multiplication
- By multiplying numerator; Reason.
- By dividing denominator; Reason.
- By removing denominator; Reason.
- When multiplier and denominator have a common factor.

Division
- By dividing numerator; Explain.
- By multiplying denominator; Explain.
- When divisor and numerator have a common factor.

Reduction of Fractions.

Define: Reduction; Lowest Terms.

Reduction
- By multiplying both terms by the same factor; Explain.
- By dividing both terms by the same divisor; Explain.
- To lowest terms.
- To given denominator; Reason.
  - Entire quantity.
  - Fraction.
- To common denominator.

Aggregation and Dissection.

Aggregation
- Of fractions having the same denominator; Give rule and reasons.
- Of an entire quantity and a fraction.
- Of fractions having different denominators.

Dissection. Show when applicable; Give rule; reason.

Multiplication and Division by Fractions.

Multiplication.
- Define multiplication by a fraction.
- Give general rule; Explain reason.
- Show how general rule is applied when the multiplicand is also a fraction.

Division.
- Define division by a fraction.
- Deduce general rule.

CHAPTER IV.

SIMPLE EQUATIONS.

Definitions.

77. *Def.* An **equation** is a statement, in the language of algebra, that two expressions are equal.

Def. The two equal expressions are called **members** of the equation.

78. *Def.* An **identical equation** is one which is true for all values of the algebraic symbols which enter into it, or which has numbers only for its members.

EXAMPLES. The equations

$$14 + 9 = 29 - 6,$$
$$(5 + 13) - (3 \times 4) - 6 = 0,$$

which contain no algebraic symbols, are identical equations. So also are the equations

$$x = x,$$
$$x - x = 0,$$
$$7(x + y) = 7x + 7y,$$

because they are necessarily true, whatever values we assign to x and y.

REMARK. All the equations used in the preceding chapters to express the relations of algebraic quantities are identical ones, because they are true for all values of these quantities.

79. *Def.* An **equation of condition** is one which can be true only when the algebraic symbols are equal to certain quantities, or have certain relations among themselves.

EXAMPLE. The equation

$$x + 6 = 22$$

can be true only when x is equal to 16, and is therefore an equation of condition.

REMARK. In an equation of condition, some of the quantities may be supposed to be known and others to be unknown.

80. *Def.* To **solve** an equation means to find such numbers or algebraic expressions as, being substituted for the unknown quantity, will render the equation identically true.

Any such value of the unknown quantity is called a **root** of the equation.

EXAMPLE 1. The number 3 is a root of the equation

$$2x^2 - 18 = 0,$$

because when we put 3 in place of x the equation is satisfied identically. Prove this. The number -3 is also a root.

An algebraic equation is solved by performing such similar operations upon its two members that the unknown quantity shall finally stand alone as one member of an equation.

REMARK. It is common in Elementary Algebra to represent unknown quantities by the last letters of the alphabet, and quantities supposed to be known by the first letters. But this is not at all necessary, and the student should accustom himself to regard any symbol as an unknown quantity.

Axioms.

81. *Def.* An **axiom** is a proposition which is taken for granted, in order that we may, by it, prove some other proposition.

Equations are solved by operations founded upon the following axioms, which are self-evident, and so need no proof.

Ax. I. If equal quantities be added to the two members of an equation, the members will still be equal.

Ax. II. If equal quantities be subtracted from the two members of an equation, they will still be equal.

Ax. III. If the two members be multiplied by equal factors, they will still be equal.

Ax. IV. If the two members be divided by equal divisors (the divisors being different from zero), they will still be equal.

Ax. V. Similar roots of the two members are equal.

These axioms may be summed up in the single one,

Similar operations upon equal quantities give equal results.

Transposing Terms.

82. THEOREM. *Any term may be transposed from one member of an equation to the other member if its sign be changed.*

EXAMPLE 1. From the equation

$$7 + 18 = 25$$

we obtain, by transposing 18,

$$7 = 25 - 18;$$

by transposing 7,

$$18 = 25 - 7.$$

Ex. 2. From the equation

$$9 = 12 - 3$$

we obtain, by transposing 3,

$$9 + 3 = 12,$$

and from this last equation, by transposing 9,

$$3 = 12 - 9.$$

EXERCISES.

Form two equations from each of the following by transposition:

1. $16 = 9 + 7$.
2. $8 + 5 = 13$.
3. $15 = 6 + 9$.
4. $23 - 10 = 13$.
5. $14 = 20 - 6$. Ans. $14 + 6 = 20$; $6 = 20 - 14$.
6. $14 = 21 - 7$.
7. $17 - 8 = 9$.

Form as many equations as you can from:

8. $8 - 5 = 9 - 6$.
9. $19 - 7 = 15 - 3$.

83. *General Proof of Transposition.* Let us have the equation

$$a + t = b.$$

Now subtract t from both members,

$$a + t - t = b - t;$$

whence, by reduction, $\quad a = b - t.$

This equation is the same as the one from which we started, except that t has been transposed to the second member, with its sign changed from + to −.

If the equation is

$$b - t = a,$$

we may add t to both members, which would give

$$b = a + t.$$

EXERCISES IN SOLVING EQUATIONS BY TRANSPOSITION.

1. Find that number which, when 9 is added to it, will make 25.

Solution. Let us call the required number n. The problem says 9 must be added to it. Adding 9 the sum is $n + 9$.

The problem says this sum must make 25. Therefore

$$n + 9 = 25.$$

Transposing 9,

$$n = 25 - 9 = 16. \text{ Ans.}$$

NOTE. This and some of the following problems are so simple that the pupil can answer them mentally; but he should do them by algebra in order to learn methods which may be applied to more difficult problems.

2. Find that number which, when 17 is added to it, will make 30.

3. Find that number which, when 12 is subtracted from it, will leave the remainder 11.

4. Find a number which, subtracted from 16, will give 9 as the remainder.

Solution. Let x be the number. When subtracted from 16, the remainder is $16 - x$. By the conditions of the question,

$$16 - x = 9.$$

Transposing x, we have

$$16 = 9 + x.$$

Transposing 9,

$$16 - 9 = x,$$

whence

$$x = 7. \text{ Ans.}$$

5. Find a number which being subtracted from 15, the remainder shall be 6.

6. If a number be diminished by 6, and the remainder multiplied by 3, the product shall be double the number. What is the number?

Solution. Let r be the number. Diminishing it by 6 the remainder is $r - 6$. Multiplying this remainder by 3

the product is $3r - 18$ (§ 38). By the condition of the problem this product is equal to $2r$. Therefore

$$3r - 18 = 2r.$$

Transposing 18,

$$3r = 2r + 18.$$

Transposing $2r$,

$$3r - 2r = 18,$$

or, by reduction,

$$r = 18. \text{ Ans.}$$

Proof. $18 - 6 = 12$; $12 \times 3 = 36$, which is twice 18.

Note. The student should prove all his answers by showing that they fulfil the conditions of the problem.

7. If 4 be subtracted from a number, and the remainder multiplied by 4, the product will be three times the number. Find the number.

8. A baker started out with x loaves of bread. He sold all but 8 of them at 5 cents each, and then had as much money as if he had sold them all at 4 cents each. What was his number x?

9. One baker started out with y loaves, and another with 9 loaves less. The first sold all his at 5 cents each, and the other sold all his at 6 cents each and realized an equal amount. How many loaves had each?

10. A huckster bought a lot of turkeys at \$1 each. Nine were spoilt, and he sold the remainder at \$2 each and made a profit of \$7. How many did he buy?

Solution. Let x be the number he bought. At \$1 each they cost him x dollars. 9 being spoilt he had $x - 9$ to sell. At \$2 each the amount was $2x - 18$ dollars. Because he made a profit of \$7 this amount must, by the conditions, be \$7 more than the cost; that is \$7 more than \$$x$. Therefore

$$2x - 18 = x + 7.$$

Transposing x and 18,

$$2x - x = 7 + 18,$$

or

$$x = 25. \text{ Ans.}$$

Proof. $25 - 9 = 16$; $16 \times 2 = 32$, which is the same as $25 + 7$.

Division of Equations.

84. From the equation

$$12 - 8 = 4$$

we obtain, by dividing by 2,

$$6 - 4 = 2,$$

and, by dividing by 2,

$$3 - 2 = 1.$$

EXERCISES.

Form as many more equations as you can from the following by division, and see if the results are true:

1. $24 - 16 = 8.$
2. $36 - 24 = 12.$
3. $72 - 45 = 27.$
4. $84 - 70 = 35 - 21.$

85. From the equation

$$3x = 24$$

we obtain, by dividing by 3,

$$x = 8.$$

Proof. Putting 8 for x in the given equation, we have

$$3 \cdot 8 = 24,$$

which equation is identically true.

If we have the equation

$$ax = m,$$

we find, by dividing by a,

$$x = \frac{m}{a}.$$

Hence, to solve an equation in which one member is known and the other member is the unknown quantity multiplied by a coefficient:

RULE. *Divide both members by the coefficient of the unknown quantity.*

EXERCISES.

Solve the following equations by reduction and transposition:

1. $a + x = b - 2x.$

Solution. Transposing a and $2x$,

$$x + 2x = b - a,$$

or

$$3x = b - a.$$

Dividing by 3,
$$x = \frac{b - a}{3}.$$

2. $7 + 3x = 37 - 2x$.
3. $5(x - 3) = x + 12$.
4. $6(x + 5) = 10x$.
5. $3(8 - x) = 23 + 2x$.

Solution of 5. $24 - 3x = 23 + 2x$.

Transposing $3x$,
$$24 = 23 + 2x + 3x = 23 + 5x.$$
Transposing 23,
$$24 - 23 = 5x,$$
or
$$1 = 5x.$$
Dividing by 5,
$$\frac{1}{5} = x, \quad \text{or} \quad x = \frac{1}{5}. \quad \text{Ans.}$$

6. $7(9 - x) = 5(11 - x)$.
7. $8(2 - n) = 5(n - 2)$.
8. $4(1 - x) = 5(2 - x)$.
9. $x + a = 2x + b$.
10. $3x - 21 + 4x - 7 = 0$.
11. $ax - b = 0$.
12. $bx = c$.
13. $ax = a + b$.
14. $(a + b)x = a - b$.
15. $mx + 1 = nx + 2$.

16. A man divided 42 apples between two boys, giving A twice as many as B. How many did each get?

Solution. Let us call n the number of apples B got. Then, because A got twice as many, he got $2n$. So, by adding,

$$\begin{aligned} \text{B's apples} &= n \\ \text{A's apples} &= 2n \\ \hline \text{Both together got } &3n \end{aligned}$$

The condition is that this number shall be 42. Hence
$$3n = 42;$$
and dividing by 3,
$$n = 14.$$
Therefore
$$\text{A's share} = 2n = 28;$$
$$\text{B's share} = n = 14.$$

17. A, B and C had between them \$78. B had twice as much as A, and C had three times as much as A. How much had each?

18. A drover bought a flock of sheep at \$3 per head. After losing 40 he sold the remainder at \$5 per head and gained \$1050. How many did he buy?

19. A man made a journey of 244 miles in 3 days, going 10 miles less on the second day than on the first, and 12 miles less on the third day than on the second. How far did he go each day?

NOTE. If we call x the distance made on the first day, that on the second will be $x - 10$ and that on the third $x - 10 - 12 = x - 22$.

20. In dividing a profit of $2500 between two partners, A got twice as much as B and $100 more. What was the share of each?

21. A huckster had 50 apples, some good and others bad. He sold the good at 5 cents each and the bad at 2 cents each, realizing $1.78 in all. How many apples of each kind had he?

Method of Solution. Let us call g the number of good apples. Then, because the good and bad together numbered 50, the bad alone numbered $50 - g$. So the sums realized from sales were:

g good apples at 5 cents each..	$5g$
$50 - g$ bad ones at 2 cents each	$100 - 2g$
Total amount received	$100 + 3g$

This expression is to be equated to the amount realized, namely, 178 cents. By solving the equation thus formed, we shall find

$$g = 26 = \text{number of good apples;}$$
$$50 - g = 24 = \text{number of bad apples.}$$
$$26 \times 5 + 24 \times 2 = 130 + 48 = 178. \quad \text{Proof.}$$

22. A man had 45 pieces of coin, some 3-cent pieces and the remainder 10-cent pieces, amounting in all to $1.84. How many pieces of each kind were there?

23. A train made a journey of 374 miles in 13 hours, going part of the time at the rate of 25 miles an hour and the remainder of the time at the rate of 32 miles an hour. How many hours did it run at each rate of speed?

NOTE. If we call x the number of hours it ran at one rate, $13 - x$ will be the number of hours it ran at the other rate.

24. A man made a journey of 112 miles in 4 days, going 4 miles farther each day than he did the day before. How far did he go on each day?

NOTE. Take the distance he went on the first day as the unknown quantity.

25. Divide 100 into two parts such that three times the one part shall be equal to twice the other.

Multiplication of Equations.

86. *Clearing of Fractions.* The operation of multiplication is usually performed in order to clear the equation of fractions.

To clear an equation of fractions:

FIRST METHOD. *Multiply its members by the least common multiple of all its denominators.*

SECOND METHOD. *Multiply its members by each of the denominators in succession.*

REMARK 1. Sometimes the one and sometimes the other of these methods is the more convenient.

REM. 2. The operation of clearing of fractions is similar to that of reducing fractions to a common denominator.

EXAMPLE OF FIRST METHOD. Clear of fractions the equation

$$\frac{x}{4}+\frac{x}{6}+\frac{x}{8}=26.$$

Here 24 is the least common multiple of the denominators. Multiplying each term by it, we have (§ 68)

$$6x+4x+3x=624,$$

or

$$13x=624.$$

EXAMPLE OF SECOND METHOD.

$$\frac{x}{a}+\frac{x}{b}=\frac{1}{ab}.$$

Multiplying by a, $\quad x+\frac{ax}{b}=\frac{1}{b}.$

Multiplying by b, $\quad bx+ax=1,$

or $\quad (a+b)x=1.$

EXERCISES.

Clear of fractions and solve the following equations:

1. $\frac{x}{2}-7=\frac{x}{4}.$

2. $\frac{1}{3}-2x=3x.$

3. $\frac{x-1}{2}=\frac{x+1}{3}.$

4. $3+\frac{5}{x}=\frac{1}{4}+\frac{7}{x}.$

5. $\frac{1}{x}+\frac{2}{x}+\frac{3}{x}=1.$

6. $\frac{a}{x}+\frac{b}{x}+\frac{c}{x}=A.$

7. $\frac{1}{2}(1-x)+\frac{1}{3}(1+x)=4x.$ 8. $\frac{x+1}{2}+\frac{x+2}{3}+\frac{x+3}{4}=0.$

9. $\frac{2x+1}{5}+\frac{3x+1}{4}=7x.$ 10. $\frac{x}{2}+\frac{x}{5}-\frac{x}{7}=4.$

11. $\frac{x}{a}+\frac{x}{b}-\frac{x}{c}=m.$ 12. $2 \,.\, 3x-7=0.$

13. $apx-k=0.$ 14. $\frac{1}{x}-\frac{2a}{mx}=3.$

15. $\frac{1}{x}-\frac{2 \,.\, 3}{5x}=3.$ 16. $\frac{4x-1}{3}-\frac{5x-1}{4}=3 \,.\, 4.$

17. $\frac{bx-1}{a}-\frac{cx-1}{b}=ab.$ 18. $\frac{x+7}{4}-1=\frac{x-4}{7}+1.$

19. $y-\frac{32-y}{8}=\frac{1}{7}y+7.$ 20. $4x+12x=4 \,.\, 12.$

Problems Leading to Simple Equations.

To solve a problem, we have *first* to state the conditions of the problem in algebraic language, and *second* to solve the equation resulting from such statement.

We have already shown how to solve the equation, but for the statement no general rule can be laid down. The following precepts will, however, serve as a guide to the beginner, who must trust to practice to acquire skill in solving problems.

1. Study the problem carefully to see what is the unknown quantity required to be found. Sometimes there are several, only one of which need be taken.

2. Represent this quantity by x, y, or any other letter or symbol whatever.*

3. Perform on and with this symbol the operations described in the problem (as in Chapter I., § 23).

4. Express the conditions, stated or implied, in the problem by means of an equation.

5. Solve the equation by the methods already explained in the last four sections.

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* Any symbol may be used. In the early history of algebra the unknown quantity was called the *thing*—in Italian *cosa*, which for brevity was written *co*.

PROBLEMS FOR PRACTICE.

1. A man asked a shepherd how many sheep he had. He replied: If you add 32 to the number of my sheep and multiply the sum by 9, the product will be 27 times the number of my sheep.

2. Another shepherd answered: If you subtract 32 from my sheep and multiply the remainder by 6, you will have double the number.

3. A baker said: If you divide the number of my loaves by 6 and add 60 to the quotient, the sum will be the number of my loaves.

4. A man sets out upon a journey from one city to another. The first day he travels one half the distance between the cities; the next day he travels one third the distance between the cities, and then finds he has still 12 miles to go. Find the distance between the cities.

Solution. Let d = the distance between the cities; then

$\frac{d}{2}$ = the distance travelled the first day,

$\frac{d}{3}$ = " " " " second day,

and these two amounts *plus* 12 miles must, by the conditions of the problem, equal the whole distance d; that is, in algebraic language the conditions of the problem are stated in the following equation:

$$\frac{d}{2}+\frac{d}{3}+12=d.$$

To solve this equation multiply each member by 6, whence

$$3d+2d+72=6d.$$

Transposing, $72=6d-3d-2d;$

and uniting terms, $72=d.$

Hence 72 miles is the required distance between the cities.

Proof. Distance travelled the first day $=\frac{72}{2}=36$ miles;

" " second " $=\frac{72}{3}=24$ miles;

" " third " $=12$ miles.

Whole distance travelled $=72$ miles.

5. Another said: If you add 9 to three times the number of my loaves and divide the sum by 9, the quotient will be 38.

6. A third baker said: If you add 136 to twice the number of my loaves and divide the sum by the number of my loaves, the quotient will be 10.

7. If you add 41 to a number and divide the sum by the number *minus* 15, the quotient will be 8. What is the number? Ans. 23.

8. If you add 96 to 6 times a number and divide the sum by twice the number *minus* 52, the quotient will be 17. Find the number. Ans. 35.

9. If you subtract 3 times a certain number from 480 and divide the difference by the number *minus* 18, the quotient will be 139. Find the number. Ans. 21.

10. Find a number such that if we divide it by 6, add 7 to the quotient, and multiply the sum by two, the product will be the number itself. Ans. 21.

11. From the following condition find the age at which Sir Isaac Newton died: If you take one third his age, one fourth his age, and one sixth his age, then add them together, and add 105 to the sum, you will get just double his age. Ans. 84.

12. If you add 5 to the year in which Sir Isaac Newton was born, then divide the sum by 9 and add 638 to the quotient, the sum will be just half the number of the year. What was the year? Ans. 1642.

13. If you take the year in which John Hancock died, add 16 to it, then take $\frac{1}{3}$, $\frac{1}{9}$, and $\frac{1}{27}$ of the sum, the sum of these three quotients will be 871. Find the year. Ans. 1793.

14. If you subtract the age of John Hancock from 100, divide the difference by 4, and add the quotient to one half his age, the sum will be 17 years less than his age. Find his age. Ans. 56.

15. If you take the population of the District of Columbia in 1870, divide it by 300, subtract 400 from the quotient, and multiply the remainder by 7, the product will be 273. What was the population? Ans. 131,700.

16. If you divide my age 10 years hence by my age 21 years ago, the quotient will be 2. What is my present age?

17. Divide \$200 among three persons, A, B and C, so that B shall have \$25 more than A, and C \$18 less than A and B together.

SOLUTION. This question differs from those preceding in having three known quantities. We therefore show how to solve it. Let us put y for A's share. Then B's share will be $y + 25$, and the share of A and B together will be $y + y + 25$; that is, $2y + 25$. C's share is said to be \$18 less than this sum; it is therefore $2y + 7$. We now add up the shares:

A's share is	y
B's " "	$y + 25$
C's " "	$2y + 7$
Sum of all,	$4y + 32$

Now by the conditions of the problem this sum must be \$200. Hence we put it equal to 200 and solve the equation. We shall thus find $y = 42$. Therefore we have for the answer:

A's share =	y =	42
B's " =	$y + 25$ =	67
C's " =	$109 - 18$ =	91
Total,		\$200. *Proof.*

18. A father left \$7050 to be divided among five children, directing that the eldest should have \$200 more than the second, the second \$200 more than the third, and so on to the youngest. What was the share of each?

Ans. \$1810, \$1610, \$1410, \$1210, \$1010.

19. A is 15 years older than B, and in 18 years A will be just twice as old as B is now. What are their ages?

Ans. 48, 33.

20. Of three brothers the youngest is 4 years younger than the second, and the eldest is as old as the other two together. In 10 years from now the sum of their ages would be 98. What are their ages now? Ans. 15, 19, 34.

21. An uncle left his property to two nephews, the elder to receive \$100 more than the younger. But if a certain cousin was still living, the elder was to give her one fourth of his share, and the younger one sixth of his. The cousin was living, and got \$1275. How much did the uncle leave, and what was the share of each heir? Ans. \$3000, \$3100.

22. The head of a fish is 9 inches long, the tail is as long as the head and half the body, and the body is as long as the

head and tail together. What is the whole length of the fish? Ans. 6 feet.

23. Divide the number 104 into two such parts that one eighth of the greater part shall be equal to one fifth of the lesser. Ans. 64, 40.

NOTE. If one part be x, the other will be $104 - x$.

24. Divide 188 into two such parts that the fourth of one part may exceed the eighth of the other by 18.

25. Two gamblers, A and B, engage in play. When they begin A has $200 and B has $152. When they finish A has three times as much as B. How much did he win?

26. A father has five sons each of whom is 5 years older than his next younger brother, and the eldest is five times as old as the youngest. What are their ages?

27. A father is now 4 times as old as his son, but in four years he will only be 3 times as old. How old is each?

28. The sum of $345 was raised by A, B and C together. B contributed twice as much as A and $15 more, and C three times as much as B and $15 more. How much did each raise?

29. Divide the number 97 into two parts such that twice the one part added to three times the other shall be 254.

Ans. 37, 60.

30. A father divided his estate among four sons, directing that the youngest should receive $\frac{1}{8}$ of the whole; the next, $1000 more; the next, as much as these two together; and the eldest, what was left. The share of the eldest was $5000. What was the value of the estate, and the shares of the three younger? Ans. $14,000; $1750, $2750, $4500.

31. An almoner divided $86 among 59 people, giving the barefoot ones $1 each and the others $1.75 each. How many barefoot ones were there?

32. An almoner divided $75 among a crowd of people, giving one third of them 50 cents each, one fourth $1 each, and the remainder $1.50 each. How many people were there?

33. A father, in making his will, directs that the second of his three sons shall have half as much again as the young-

est, and the eldest one third more than the second. The eldest receives $2250 more than the youngest. What was the share of each?

34. Divide the number 144 into four parts such that the first part divided by 3, the second multiplied by 3, the third diminished by 3, and the fourth increased by 3 shall all four be equal to each other.

Call x the quantity to which these four results are equal. Then the part which divided by 3 will make x must be $3x$, the second part must be $\frac{x}{3}$, the third $x + 3$, and the fourth $x - 3$. To form the equation note the sum of all the parts is 144, which gives the equation to be solved.

Ans. 81, 9, 30, 24.

35. Divide the number 100 into four parts such that the first being multiplied by 4, the second divided by 4, the third increased by 4, and the fourth diminished by 4 shall give equal results.

36. A man making a journey went on the first day one third of the distance and 36 miles more; on the second, one third the remaining distance and 36 miles more, which just brought him to his journey's end. What was the length of the journey?

37. What number of apples was divided among three people when the first was given half the apples and half an apple more, the second half of what remained and half an apple more, and the third half of what then remained and half an apple more, which emptied the basket?

38. In the division of an estate between A, B, C and D, A got one tenth of the whole, B got half as much as A and $2505 more, C got half as much as B and $2505 more, and D got $4050 dollars. What was the amount of the estate?

39. A person journeying to a distant town travelled on the first day half way, wanting 20 miles; on the second day half the remaining distance, wanting 5 miles; and on the third day half the distance still remaining and 10 miles more, when he still had 12 miles to travel on the fourth day. How long was his whole journey?

40. A trader having made a profit of r per cent on his capital found that capital and profits together amounted to a dollars. What was his capital?

NOTE. By the definition of percentage, r per cent of a sum means r hundredths of that sum. Hence r per cent is found by multiplying by $\frac{r}{100}$. If the capital is c, r per cent of it is $\frac{rc}{100}$, and when this is added to the capital the sum will be $c + \frac{rc}{100}$.

41. A trader having increased his capital by 8 per cent found that it amounted to $4320. How much was it at first?

42. A merchant increased his capital by 8 per cent the first year and increased that increased capital by 12 per cent the second year, when the total amounted to m dollars. How much had he at first?

43. Of three casks the second contained 15 per cent more than the first, and the third 20 per cent more than the second. The first and third together contained 119 gallons. How much did the second contain?

44. A man investing a sum of money got 5 per cent interest on it the first year, which he added to his principal, and then got 4 per cent on the amount for the second year. At the end of the second year he found interest for the two years to amount to $2457. How much did he invest?

45. Of two men A and B, A had in money 25 per cent more than B. B having received $225 then had 25 per cent more than A. How much had each at first?

46. The perimeter of a triangle measures 320 yards. The first side is 40 yards longer than the base, and the second side is half the sum of the base and first side. What is the length of each side?

NOTE. By the perimeter of a triangle is meant the sum of its three sides. The base is any one of the three sides.

47. The perimeter of a triangle measures p feet. The first side is m feet longer than the base, and the remaining side is m feet longer than the first side. What are the lengths of the three sides?

48. A pedestrian made a journey of 125 miles in 5 days, going 3 miles less on each day than he did on the day preced-

ing. How far did he go on the first day? How far on the last day?

49. A man bought 3 works each containing a volumes, and 2 works each containing b volumes, for x dollars. What was the price of each volume?

50. A man bought a work of m volumes for x dollars. How much would k volumes cost at the same price per volume?

51. A line 64 feet long is divided into three parts such that the second part is one fourth longer than the first, and the third part one fourth shorter than the first and second together. What is the length of each part?

52. A factory employed men at \$2 per day, 38 more women than men, paying them \$1 per day, and 35 more boys than women, paying the boys 60 cents per day. The daily wages amounted to \$197. How many operatives were there of each class?

53. In another factory there were one third more women than men, and one third more boys than men and women together. The wages were: men \$1.50, women \$1, boys 50 cents,—making a sum total of \$276.50. How many operatives were there of each class?

54. Four men having received a sum of D dollars to be divided between them, the first got $\frac{1}{n}$ of the whole, and the second half as much as the first and x dollars more; the remainder being divided equally between the third and fourth. How much did each get?

55. If from a sum of x dollars I take one third, then one half of what is left, and then one third of what is still left, how much remains?

56. Having an equation

$$ax = b,$$

I want to put for a and b two numbers whose sum shall be 20, and which shall give $\frac{3}{7}$ for the value of x. What numbers shall I use for a and b?

57. In another equation of the same form I want b to be greater than a by 12, and the value of x to be 2. What numbers shall I use for a and b?

58. In the fraction

$$\frac{m+x}{3m+x}$$

I want to put for x such a quantity that the value of the fraction shall be $\frac{3}{4}$. What value of x shall I use?

59. What must be the value of m in order that the sum of the two fractions $\frac{3}{m+1}$ and $\frac{4}{m+1}$ may be $\frac{1}{2}$?

60. Out of a bin of wheat one man took one fourth of the wheat and 3 bushels more, and a second took one third of what was left and 5 bushels more. What he left was 5 bushels more than the first man took. How much wheat was in the bin, and how much did each take?

Memoranda and Exercises for Review.

Define: Equation; Identical equation; Equation of condition; Solving an equation; Root; Axiom.

Addition and Subtraction.	Axioms of addition and subtraction. Rule for transposition; Give examples. Prove the general rule. Write an equation which may be solved by simple transposition and aggregation of terms.
Division.	Axiom of division. Solution of an equation by division; Rule. Write an equation with four terms which can be solved by transposition and division.
Multiplication.	Axiom of multiplication. When multiplication is required. Two methods; Explain both.

SECOND COURSE.

ALGEBRA TO QUADRATIC EQUATIONS.

CHAPTER I.

THEORY OF ALGEBRAIC SIGNS.

Use of Positive and Negative Signs.

88. *Opposite Directions of Measurement.* Most of the quantities we have to express in algebraic language may be measured in two opposite directions.

EXAMPLES. *Time* may be either time *before* or time *after*.

Distance on a straight road may be measured from any point in two opposite directions. If the road is east and west, one direction will be *west* and the other *east*.

The *scale of a thermometer* is divided so as to measure from zero in two opposite directions.

89. *Positive and Negative Measures.* In order to measure such quantities on a uniform system, the numbers of algebra are considered to increase from 0 in two opposite directions. Those in one direction are called **positive**; those in the other direction, **negative.**

Positive numbers are distinguished by the sign $+$, negative ones by the sign $-$.

If a positive number measures years after Christ, a negative one will mean years before Christ.

If a positive number is used to measure toward the right, a negative one will measure toward the left.

If a positive number measures weight, the negative one will imply lightness, or tendency to rise from the earth.

If a positive number measures property, or credit, the negative one will imply debt.

90. The series of algebraic numbers will therefore be considered as arranged in the following way, the series going out to infinity in both directions:

Negative Direction.	Positive Direction.
Before.	After.
Downward.	Upward.
Debt.	Credit.
etc.	etc.

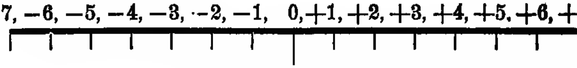
etc. $-5, -4, -3, -2, -1, 0, +1, +2, +3, +4, +5,$ etc.

91. *Choice of Direction.* It matters not which direction we take as the positive one, so long as we take the opposite one as negative.

If we take time *before* as positive, time *after* will be negative; if we take *west* as the positive direction, *east* will be the negative; if we take *debt* as positive, *credit* will be negative.

92. *The Scale of Numbers.* Positive and negative numbers may be conceived to measure distances from a fixed point on a straight line, extending indefinitely in both directions, the distances on one side being positive, on the other side negative, as in the following diagram: *

etc. $-7, -6, -5, -4, -3, -2, -1, \ 0, +1, +2, +3, +4, +5, +6, +7,$ etc.



Positive direction.

$\longrightarrow$

Negative direction.

$\longleftarrow$

In the above *scale of numbers* the *positive* direction is from *left to right* and the *negative* direction from *right to left*, as shown by the arrows; and the distance between any two consecutive numbers is considered a *unit* or *unit step*.

* The student should copy this scale of numbers, and have it before him in studying the present chapter.

93. *Def.* In the scale of numbers any number which lies in a positive direction from another is called **algebraically greater** than that other. Thus,

-2	is	algebraically	greater	than	-7;
0	"	"	"	"	-2;
5	"	"	"	"	-5.

Def. The **absolute value** of a quantity is its value without regard to its algebraic sign.

EXAMPLE. The absolute value of -3 is the same as the absolute value of $+3$; namely, 3 without any sign.

94. *Meaning of Minus.* We now have the following new definition of the minus sign:

Minus means *opposite.* A minus sign shows that the quantity before which it is placed must be taken in the opposite sense from that in which it would be taken if the sign were not there.

EXAMPLE 1. $-x$ is the opposite of x.

Ex. 2. $-(-x)$ is the opposite of $-x$; that is, it is x.

Therefore $$-(-x) = x.$$

Ex. 3. $-[-(-x)]$ is the opposite of $-(-x)$, or of x.

Hence $$-[-(-x)] = -x. \quad \text{etc. etc.}$$

EXERCISES.

What is the meaning of:

1. $-n$ years after Christ?
2. $-(-n)$ years after Christ?
3. $-n$ years before Christ?
4. $-(-n)$ years before Christ?
5. The thermometer is $-15°$ above zero?
6. The thermometer has risen $-20°$?
7. The thermometer has fallen $-(-20°)$?
8. To-day is $-25°$ colder than yesterday?
9. To-day is $-25°$ warmer than yesterday?
10. To-day is $-(-25°)$ warmer than yesterday?
11. John lives -2 miles east of William?

    Ans. John lives 2 miles west of William.
12. James lives -4 miles south of John?

13. Smith is -6 years older than Jones?
14. Jones is -5 years younger than Brown?
15. John owes the grocer $-\$5$?
16. Thomas weighs -12 pounds more than Jane?
17. Mr. Weston is $-\$1000$ richer than Mr. Brown?

Answer the following questions in algebraic language:

18. James owed William \$12 and paid him \$7. How much did he still owe? How much did William owe James?

19. If James had owed \$12 and paid \$15, how much would he still owe?

20. If he owed William a dollars and paid him b dollars, how much would he still owe? How much would William owe him?

21. The thermometer was 15° on Sunday, and next day it was 22° lower. What was it then?

22. What day is the 0th of February? The -1st? The -4th? The -31st?

Note that the 0th day is the day before the 1st day.

95. *Def.* When two quantities are numerically equal but with opposite signs, each is said to be the **negative** of the other.

EXAMPLES. $-a$ is the negative of a.
a is the negative of $-a$.
$x-y$ is the negative of $y-x$.

EXERCISES.

What is the negative of:

1. $3a$? 2. $-2b$? 3. $x-a-b$?
4. $-2y+3a-b$? 5. $-(a-b)$?

96. *Lines to represent Numbers.* A number may be represented to the eye by drawing a line long enough to reach from the zero point to the number on any fixed scale.

To fix the scale a length must be assumed as a unit, and a direction must be chosen as positive. The line must then have as many units as the number to be represented.

A negative number is drawn in the opposite direction from a positive one.

EXAMPLES. If we take this length ——— as the unit,
$+3$ is this line o—————————
-2 this line ——————o
and -1 this line ———o

The zero point is always the beginning of the line.

EXERCISES.

Taking any unit you please, draw, by the eye, lines to represent:

1. $+2$. 2. $+5$. 3. -4. 4. -3. 5. -1.

Algebraic Addition.

97. Algebraic addition is the operation of combining quantities according to their algebraic signs, and is performed by taking the difference between the sums of the positive and negative quantities and prefixing the sign of the greater sum.

Def. The result of algebraic addition is called the **algebraic sum.**

EXAMPLE. The algebraic sum of $4+5-7-8+2$ is
$$+11-15=-4.$$

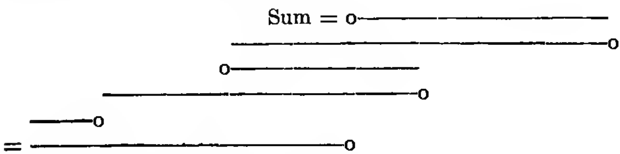
98. Algebraic addition is represented on the scale of numbers by measuring off the positive quantities in the positive direction, and the negative quantities in the opposite direction, commencing each measure at the end of the one preceding.

EXAMPLE. To represent the algebraic sum
$$4-6+3-5-1,$$
we start from 0 and measure off 4 unit steps to the right, which takes us to $+4$. Then we measure 6 from $+4$ toward the left, which brings us to -2. Then 3 to the right brings us to $+1$. Then 5 to the left brings us to -4. Then 1 to the left brings us to -5, which is the algebraic sum. Thus
$$4-6+3-5-1=-5.$$

99. *Formation of Algebraic Sums.* We see from the above that the algebraic sum of several numbers may be

formed by putting the lines which represent them end to end in their proper directions, the beginning, or zero point, of each one being at the end of the preceding one.

The preceding example $(+4-6+3-5-1)$ is then represented in this way:



Examples of Algebraic Sums. Three traders agree to divide their profits and losses equally. The first year A gained \$3000, B \$4000 and C \$8000. The algebraic sum of the profits to be divided was therefore

$$\$3000 + \$4000 + \$8000 = \$15{,}000.$$

Each man's share = \$5000.

Next year A gained \$4000 and B \$5000, while C lost \$3000. The algebraic sum to be divided was therefore

$$\$4000 + \$5000 - \$3000 = \$6000.$$

Each man's share = + \$2000.

The third year A gained \$2000, B lost \$1000 and C lost \$4000. The algebraic sum of their profits was therefore

$$\$2000 - \$1000 - \$4000 = -\$3000.$$

Therefore each man got − \$1000; that is, he suffered a loss of \$1000.

EXERCISES.

Draw a scale of numbers (§ 92) from -8 to $+8$, and then form the following algebraic sums by lines under the scale:

1. $-4+6-3+5+1.$ 2. $1-2+3-4.$
3. $5-3-3-3-3.$ 4. $-4-4+3+4+5.$

5. The temperature at 9 points scattered equally over the country is $+15°$, $+20°$, $-8°$, $-13°$, $-2°$, $+7°$, $-15°$, $-10°$ and $-12°$. What is the mean temperature?

NOTE. The mean of any series of numbers is obtained by dividing their algebraic sum by the number of terms.

6. A merchant in 5 successive years lost \$3000, made \$2000, made \$8000, lost \$1000 and made \$3000. What was his average annual profit?

Algebraic Subtraction.

100. Subtraction in algebra consists in expressing the *algebraic difference* between two quantities.

Def. The **algebraic difference** of two quantities is their number of units apart on the scale of numbers.

EXERCISES.

What is the algebraic difference between:

1. -3 and $+5$? Ans. 8.
2. $+3$ and $+5$?
3. -7 and -3?
4. $+5$ and -3?
5. -9 and $+9$?

101. *Sign of Remainder.* The sign of the algebraic difference is shown by the direction *from* the subtrahend *to* the minuend.

EXAMPLES. *From* -8 *to* -3 is $+ \therefore -3-(-8)=+5$.
From -3 *to* -8 is $- \therefore -8-(-3)=-5$.

REMARK. When we subtract a lesser positive quantity from a greater one, this rule gives a positive remainder, so that the result is the same as in arithmetic. But the algebraic process takes account of cases which arithmetic does not; for example, those where the subtrahend is greater than the minuend. This reversal of the case is indicated by the negative sign of the difference.

EXERCISES.

Give the values of the three quantities $a+b$, $a-b$ and $b-a$ in the following cases:

1. When $a=+13$ and $b=+8$.
2. When $a=+13$ and $b=-8$.
3. When $a=-13$ and $b=+8$.
4. When $a=-13$ and $b=-8$.
5. Of two bankrupts, A and B, A owes $5000 more than all he possesses, and B owes $3000 more. Which is the richer, and how much richer is he?
6. In this case, what is the sum of their possessions?
7. Of two merchants, A and B, A made $2000 and then lost $3000; B lost $2000 and then made $7000. How much more was A worth than B? How much more was B worth than A?

8. During two years A gained a dollars and afterward lost b dollars; B first gained x dollars and then lost y dollars. Express how much A was worth more than B.

Rule of Signs in Multiplication.

102. I. In multiplying a line by a positive factor, we leave its direction unchanged.

II. In multiplying by a negative factor, we change the direction of the line.

Illustration. Suppose the quantity a to represent a length of one centimetre from the zero point toward the right on the scale of § 92. Then we shall have

$a =$ this line 

The products of this line by the factors from $+2$ to -3 will be:

$a \times 2$ . . . .  $= +2a$.

$a \times 1$ . . . .  $= a$.

$a \times 0$ . . . . 0 $= 0$.

$a \times -1$ . . . .  $= -a$.

$a \times -2$ . . . .  $= -2a$.

$a \times -3$ . .  $= -3a$.

We shall also have

$-a =$ this line 

The products of this line by the factors from $+3$ to -2 will be:

$-a \times 3$ . .  $= -3a$.

$-a \times 2$ . . . .  $= -2a$.

$-a \times 1$ . . . .  $= -a$.

$-a \times 0$ . . . . 0 $= 0$.

$-a \times -1$ . . . .  $= +a$.

$-a \times -2$ . . . .  $= +2a$.

The preceding result may be expressed thus:

In multiplication LIKE *signs give* PLUS.

UNLIKE *signs give* MINUS.

103. When there are three or more negative factors, the product of two of them will be $+$, the third will change the sign to $-$, the fourth will change this $-$ to $+$ again, and so on. Hence:

The product of an EVEN *number of negative factors is positive.*

The product of an ODD *number of negative factors is negative.*

EXERCISES.

Form the following products by the method of § 37, and assign the proper signs:

1. $-2 \times 3 \times -4$.
2. $3 \times -5 \times -6 \times -a$.
3. $-a \times -a^2x \times -a^2x^3$.
4. $m \times cm^2 \times -c^2x$.
5. $-2b \times 3b \times 4bc$.
6. $3b \times -5ab \times 7c$.
7. $-5 \times -6 \times -7 \times -8$.
8. $-1 \times -2 \times -3 \times a^2$.
9. $-1 \times -1 \times -1$.
10. $-1 \times -1 \times -1 \times -1$.
11. $-abc \times -ab \times -a \times x$.
12. $m \times -1 \times m \times m^2$.
13. $(-1)^2$.
14. $(-1)^3$.
15. $(-1)^4$.
16. $(-1)^8$.
17. $(-1)^9$.
18. $\frac{m}{n} \times -\frac{n}{m}$.
19. $\frac{a^2}{b^3} \times -\frac{b^2}{a^3} \times -ab \times \frac{a}{b}$.
20. $(-1)^{2n}$.

Rule of Signs in Division.

104. The rule of signs in division corresponds to that in multiplication, namely:

If dividend and divisor have the SAME *sign, the quotient is positive.*

If they have OPPOSITE *signs, the quotient is negative.*

Proof.

$+mx \div (+m) = +x$, because $+x \times (+m) = +mx$.
$+mx \div (-m) = -x$, " $-x \times (-m) = +mx$.
$-mx \div (+m) = -x$, " $-x \times (+m) = -mx$.
$-mx \div (-m) = +x$, " $+x \times (-m) = -mx$.

The condition to be fulfilled in all four of these cases is that the product, *quotient* $\times$ *divisor,* shall have the same algebraic sign as the dividend.

EXERCISES.

Express the following divisions, reducing fractions to their lowest terms:

1. $a^3 - a^2b + ab^2 - ab^3 \div + ab$. 2. $a^5 - a^3x^2 + ax^4 \div - ax$.
3. $- ab - bc - ca \div abc$. 4. $a^2b - b^2c - c^2a \div - a^2b^2c^2$.

105. *Rule of Signs in Fractions.* Since a fraction is an indicated quotient, the rule of signs corresponds to that for division. The following theorems follow from the laws of multiplication and division:

1. *If the terms are of the same sign, the fraction is positive; if of opposite signs, it is negative.*

2. *Changing the sign of either term changes the sign of the fraction.*

3. *Changing the signs of both terms leaves the fraction with its original sign.*

4. *The sign of the fraction may be changed by changing the sign written before it.*

EXAMPLE 1. $\frac{m}{n} = \frac{-m}{-n} = -\frac{-m}{n} = -\frac{m}{-n}$.

Ex. 2. $-\frac{m}{n} = -\frac{-m}{-n} = \frac{-m}{n} = \frac{m}{-n}$.

Ex. 3. $\frac{m-n}{a-b} = \frac{n-m}{b-a} = -\frac{m-n}{b-a} = -\frac{n-m}{a-b}$.

EXERCISES.

Express each of the following fractions in three other ways with respect to signs:

1. $\frac{m-n}{c}$. 2. $-\frac{m-n}{c}$. 3. $\frac{c}{m-n}$.

4. $\frac{x-y}{m-n}$. 5. $\frac{p+q}{p-q}$. 6. $\frac{p-q+r}{p+q-r}$.

Write the following fractions so that the symbols x and y shall be positive in both terms:

7. $\frac{x-a}{c-y}$. 8. $\frac{a-x}{b-y}$. 9. $\frac{a-x-b}{m+y-c}$.

10. $-\frac{x-a}{c-y}$. 11. $\frac{a-kx}{b+ky}$. 12. $-\frac{a-x-b}{m+y-c}$.

Aggregate the following fractional expressions:

13. $\dfrac{x}{m} - \dfrac{2x}{-m} + \dfrac{3x}{-m} - \dfrac{x-2a}{-m} - \dfrac{a}{m}$.

14. $\dfrac{c-y}{h} - \dfrac{c+y}{-h} + \dfrac{c-2y}{-h} - \dfrac{2c-y}{h}$.

15. $\dfrac{m-n}{a-n} - \dfrac{-m}{a-n} - \dfrac{-m-2a}{n-a} + \dfrac{2m-a}{n-a}$.

106. General Remark. It is necessary to the interpretation of an algebraic result that the positive sense or direction be defined in the case of each symbol which admits of either sign.

The understanding is that the sense in which the symbol is defined is positive.

Example. If we say, "Let t be the number of days *before*," we understand that days before are *positive* and days after *negative.*

But if we say, "Let t be the number of days *after*," the reverse is understood.

107. *Exercises in changing Algebraic Expressions into Numbers.* Compute the values of the following expressions when

$$a = 2, \qquad p = -3,$$
$$b = 7, \qquad q = -4,$$
$$c = 5, \qquad r = -8.$$

1. $a+p$. 2. $a-p$. 3. $b+q$.

4. $b-q$. 5. $a+r$. 6. $a-r$.

7. $2r-3b$. 8. $qr-cp$. 9. $(pq-ac)^a$.

10. $(cr-abp)^{\frac{ar}{q}}$. 11. $\left(\dfrac{q}{p}\right)^2$. 12. $\left(-\dfrac{q}{p}\right)^3$.

13. $\dfrac{a^c+q^{-p}}{a^c-q^{-p}}$. 14. $\dfrac{c^2r-p^2q}{q+r}$. 15. $\dfrac{pqr-abc}{pqr+abc}$.

16. $\dfrac{abq}{q-p}$. 17. $\dfrac{bcp-aqr}{aqp-bcr}$. 18. $\dfrac{p^c-c^{-p}}{p-c}$.

19. Compute the several values of the expression

$$ax^2+qx+r$$

for $\qquad x = -4,\ -3,\ -2,\ -1,\ 0,\ 1,\ 2,\ 3,\ 4.$

CHAPTER II.

OPERATIONS WITH COMPOUND EXPRESSIONS.

SECTION I. PRELIMINARY DEFINITIONS AND PRINCIPLES.

108. *Aggregate.* A polynomial enclosed between parentheses in order to be operated upon as a single symbol is called an **aggregate.**

Entire. An **entire quantity** is one which is expressed without any denominator or divisor, as 2, 3, 4, etc.; a, b, x, etc.; $2ab$, $2mp$, $ab(x-y)$, etc.

Formula. A **formula** is an algebraic expression used to show how a quantity is to be calculated.

Reciprocal. The **reciprocal** of a number is unity divided by that number. In the language of algebra,

$$\textit{Reciprocal of } N = \frac{1}{N}.$$

Function. An algebraic expression containing any symbol is called a **function** of the quantity represented by that symbol.

EXAMPLE 1. The expression $3x^2$ is a function of x.

EX. 2. The expression $\dfrac{a+x}{a-x}$ is a function of x. It is also a function of a.

Degree. The **degree** of a term *in* one or more symbols is the number of times it contains such symbols as factors.

EXAMPLES. The expression abx^2y^3 is of the fourth degree in a, b and x, because it contains these symbols four times as factors.

It is also of the third degree in x, of the fifth degree in x and y, and of the seventh degree in a, b, x and y.

The expression ab^2x^3 is of the fifth degree in b and x, because it contains b twice and x three times as a factor.

When an expression consists of several terms, its degree is that of its highest term.

Principles of Algebraic Language.

109. FIRST PRINCIPLE. *We may employ a single symbol to represent any algebraic expression whatever.*

The fact that a symbol is meant to represent an expression is indicated by the sign $\equiv$.

EXAMPLE. The statement

$$p \equiv ax + by$$

means, we write the symbol p to *represent* the expression $ax + by$.

SECOND PRINCIPLE. *Any algebraic expression may be operated with as if it were a single symbol.*

An expression thus operated on is enclosed in parentheses when necessary to avoid ambiguity.

As a consequence of this principle, an algebraic expression between parentheses may be enclosed between other parentheses, and these between others to any extent. Each order of parentheses must then be made thicker or different in form to distinguish them.

THIRD PRINCIPLE. *An identical equation will remain true when any expression or number is written in place of each symbol.*

EXERCISES.

Let us put $P \equiv a + b$;

$Q \equiv a - b$;

and $R \equiv ab$.

It is then required to substitute in the following expressions a and b in place of P, Q and R.

1. PQ. Ans. $(a + b)(a - b)$.
2. PQR^2. Ans. $(a + b)(a - b)(ab)^2$.
3. $P(P - Q)$. Ans. $(a + b)\{a + b - (a - b)\}$.

4. $P(R - P)$.
5. $P(R - Q)$.
6. $Q(P - QR)$. Ans. $(a - b)\{a + b - (a - b)ab\}$.
7. $Q(Q - PR)$.
8. $P(QR - P)$.
9. $\dfrac{P(R - Q)}{R(Q - P)}$.
10. $\dfrac{Q(R - P)}{P(R - Q)}$.

110. *Exercises in Compound Expressions.* Compute the value of the following expressions when

$$a = 6, \qquad m = -4,$$
$$b = 9, \qquad n = -5.$$

1. $\{(a - b)m + 3n\}b$. Ans. $(-3m + 3n)9 = -27$.
2. $\{m(b - n) + a(b + n)\}(b - a)$.
3. $a\{b(m + n) - a(m - n)\}$.
4. $\{a + a(b - m)\}^2 - \{m + n(m - a)\}^2$.
5. $\{(a - b)^3 + (m - n)^3\}(m + n)$.
6. $(a - b + m - n)^3(m + n)$.
7. $\{(a - m)(b - n)^2 - (a - n)(b - m)^2\}^2$.

Section II. Clearing of Compound Parentheses.

111. When expressions in parentheses are enclosed between others, they may be removed by applying the rules of §§ 32 and 33 to one pair at a time.

We may either begin with the outer ones and go inward, or begin with the inner ones and go outward.

It is common to begin with the inner ones.

EXERCISES.

1. Clear of parentheses $P - \{a - [m - (x - y)]\}$.

Solution. Beginning with the inner parentheses and applying the rule of § 33 to each pair of parentheses in succession, the expression takes the following forms:

$$\begin{aligned} &P - \{a - (m - x + y)\} \\ &= P - \{a - m + x - y\} \\ &= P - a + m - x + y. \quad \text{Ans.} \end{aligned}$$

2. $x - \{2a - x - (a - 5x) - (-3a + x)\}$.

Solution. Removing the inner parentheses, we have

$$\begin{aligned} &x - \{2a - x - a + 5x + 3a - x\} \\ &= x - 2a + x + a - 5x - 3a + x \\ &= -2x - 4a. \quad \text{Ans.} \end{aligned}$$

3. $a+\{-(a-2b)+(2a-b)-(-4a-3b)\}$.
4. $-(x-a)-(2x-3a)+(5x+7a)$.
5. $\{m-[m-(m-2m)]\}$.
6. $2mx-\{3mx+py-(5mx-2py)-(-6py+mx)\}$.
7. $-(6by+3mu)+\{-2by-(mu-3by)+5by+2mu\}$.
8. $p-\{2p-[3p-(4p-q)]\}$.
9. $a-x-\{a-x-(a+x)-(x-a)\}$.
10. $3ax-2by-\{ax-by-[ax-by-(by+ax)]\}$.
11. $x-\{2m+(5x-a-b)-(7a+2b)\}$.
12. $-(3cy+2mz)-\{2cy-(3mz-2cy)\}$.
13. $p-q-\{2p-3q-(4p+5q)+(6p-q)\}$.
14. $p+q+\{-3p-4q-(2p-7q)+(3p+4q)\}$.

Section III. Multiplication.

General Laws of Multiplication.

112. Law of Commutation. *Multiplier and multiplicand may be interchanged without altering the product.*

This law is proved for whole numbers in the following way: Form several rows of quantities, each represented by the letter a, with an equal number in each row, thus:

$$\begin{array}{cccccc} a & a & a & a & a & a \\ a & a & a & a & a & a \\ a & a & a & a & a & a \\ a & a & a & a & a & a \\ a & a & a & a & a & a \end{array}$$

Let m be the number of rows, and n the number in each row. Then counting by rows there will be

$$n \times m \text{ quantities.}$$

Counting by columns there will be

$$m \times n \text{ quantities.}$$

Therefore $\quad m \times n = n \times m$,

or $\quad nm = mn$.

Remark. This is called the law of *commutation*, from *commuto*, I exchange.

113. LAW OF ASSOCIATION. *When there are three factors, m, n and a,*

$$m(na) = (mn)a.$$

EXAMPLE. $3 \times (5 \times 8) = 3 \times 40 = 120;$
$(3 \times 5) \times 8 = 15 \times 8 = 120.$

Proof for Whole Numbers. If a in the above scheme represents a number, the sum of each row will be na. Because there are m rows, the whole sum will be $m(na)$.

But the whole number of a's is mn. Therefore

$$m(na) = (mn)a.$$

REMARK. This is called the law of *association*, because the middle factor, n, is associated first with a and then with m.

114. THE DISTRIBUTIVE LAW. *The product of a polynomial by a factor is equal to the sum of the products of each of the terms by the same factor. That is,*

$$m(p+q+r+\text{etc.}) = mp+mq+mr+\text{etc.} \qquad (1)$$

Proof for Whole Numbers. Let us write each of the quantities p, q, r, etc., m times in a horizontal line, thus:

$$\begin{array}{l}
p+p+p+\text{etc., } m \text{ times} = mp. \\
q+q+q+\text{etc., } m \text{ times} = mq. \\
r+r+r+\text{etc., } m \text{ times} = mr. \\
\qquad \text{etc.} \qquad\qquad \text{etc.} \qquad \text{etc.}
\end{array}$$

If we add up each vertical column on the left-hand side, the sum of each will be $p+q+r+$ etc., the columns being all alike.

Therefore the sum of the m columns, or of all the quantities, will be

$$m(p+q+r+\text{etc.}).$$

The first horizontal line of p's being mp, the second mq, etc., the sum of the right-hand column will be

$$mp+mq+mr+\text{etc.}$$

Since these two expressions are the sums of the same quantities, they are equal, as asserted in equation (1).

REMARK. This is called the *distributive* law, because the multiplier m is distributed to the several terms p, q, r, etc., of the polynomial.

Formation of Products.

115. *Products of Aggregates.* Expressions between parentheses may be multiplied and the parentheses removed by successive application of the principles of § 38, and of the distributive law.

EXERCISES.

1. $a\{m - n(p - q)\}$.

Solution. Applying the rule of § 38, we have

$a\{m - np + nq\} = am - anp + anq$. Ans.

2. $m\{ax - b(y - z)\}$.
3. $p\{qm - b^2(a - b)\}$.
4. $p\{p + q(p - q)\}$.
5. $m^2\{ax - 3(ax - by) + 5(a^2x - b^2y)\}$.
6. $n\{p + q[x + y(a - b)]\}$.
7. $r\{t - m[x - y(p - q)]\}$.
8. $r\{1 + r[1 + r(1 + r)]\}$.
9. $r\{1 - r[1 - r(1 - r)]\}$.
10. $a + x\{b + x[c + x(d + x)]\}$.
11. $d - x\{c - x[b - x(a - x)]\}$.
12. $a^2\{m(b - a)n - n(a - b)m\}ax^2$.
13. $pq\{a(b - c) + b(c - a) + c(a - b)\}pr$.
14. $pq\{a(b - c) + 2b(c - a) + 3c(a - b)\}pr$.
15. $\frac{m}{n}\left\{-\frac{n}{m}\left[1 - \frac{m}{n}\left(1 - \frac{n}{m}\right)\right]\right\}$.

116. *Arrangements according to Powers of a Symbol.* We may collect all the coefficients of each power of some one symbol, and affix the power to their aggregate, by the process of § 26.

Def. When a polynomial is arranged according to powers of a symbol it is called an **entire function** of that symbol.

EXAMPLE. Arrange according to powers of x,

$$x(4ax - 2b) - b(3x^2 - b^2) + (a - 4b)x - 3ax^2 + 4c.$$

Solution. Clearing of parentheses,

$$4ax^2 - 2bx - 3bx^2 + b^3 + ax - 4bx - 3ax^2 + 4c.$$

Taking the coefficients of the several powers of x, we find them to be:

Coefficients of $x^2 = 4a - 3b - 3a = a - 3b$;
Coefficients of $x = -2b + a - 4b = a - 6b$;
Term without $x = b^3 + 4c$.

Therefore the expression is equal to

$$(a - 3b)x^2 + (a - 6b)x + b^3 + 4c.$$

EXERCISES.

Arrange the following expressions according to the powers of x or other leading symbol:

1. $(a + x)x^2 - (b - x)x^3 + cx - d$.
2. $(a + bx)x + (c + dx)x^2 - a$.
3. $(mx^2 - n - p)x + (2p - q)x - ax^3 + bx^2 - cx$.

117. By combining the operations of the two preceding articles entire functions of one symbol may be expressed as entire functions of another symbol.

EXERCISES.

Arrange the following polynomials according to powers of x:

1. $(2x - 3x^2)y^3 - (x + 2x^3)y^2 + (3x^4 - 2x^2 - a)y$.
2. $(ax^2 + bx^3)y + y^2\{ax^2 - (3a + y)x\} - 2xy^3$.
3. $(a + a^2xy + x^3)a^2 + (b + cx + mx^2)a + 2cx - nx^2$.
4. $a\{m(x^2 - y) - n(x^3 - y^2)\} + b(y^3 + y^2x + yx^2 + x^3)$.
5. $(a^3 + 2b^2x + 2cx^2 + x^3)xy - m(b^3 + 2c^2x - 2dx^2)$.

Multiplication of Polynomials by Polynomials.

118. Let us consider the product

$$(a + b)(p + q + r).$$

Regarding $a + b$ as a single symbol, the product by §114 is

$$(a + b)p + (a + b)q + (a + b)r.$$

But

$$(a + b)p = ap + bp;$$
$$(a + b)q = aq + bq;$$
$$(a + b)r = ar + br.$$

Therefore the product is

$$ap + bp + aq + bq + ar + br.$$

It would have been shorter to first clear the parentheses from $(a+b)$, putting the product into the form

$$a(p+q+r)+b(p+q+r).$$

Clearing the parentheses again, we should get the same result as before.

We have therefore the following rule for multiplying one polynomial by another.

119. RULE. *Multiply each term of the multiplicand by each term of the multiplier, and add the products with their proper algebraic signs.*

EXERCISES.

1. $(m-n)(p-q)$.

Solution. $(m-n)p = mp - np;$
$(m-n)\times(-q) = -mq+nq.$
$\therefore$ product $= mp - np - mq + nq.$

2. $(a-b)(x-y+z)$.
3. $(a+b)(x+y-z)$.
4. $(a+b)(x+y)+(a-b)(x-y)$.
5. $(m-n)(x+y)-(m+n)(x-y)$.
6. $(p+q)(ax-by)+(p-q)(ax+by)$.
7. $(m-q)(mx+qy)+(m-q)(qx-my)$.
8. $(mx+ny)(my-nx)+(mx-ny)(my+nx)$.
9. $(2a-b)(2a-3b+4c)$.
10. $(x-1)(x-2)$.
11. $(x-a)(x-b)-(x+a)(x+b)$.
12. $(2a-2)(a+1)$.
13. $(1+m)(2-m)$.
14. $\left(\frac{m}{n}+\frac{n}{m}\right)\left(1+\frac{m}{n}+\frac{n}{m}\right)$.
15. $\left(\frac{a}{b}+1\right)\left(\frac{b}{a}-1\right)$.
16. $\left(\frac{m^2}{n^2}-\frac{m}{n}\right)\left(\frac{n^2}{m^2}-\frac{n}{m}\right)$.
17. $(a^2+a+1)(a^2-a+1)$.
18. $(a^2+ab+b^2)(a^2-ab+b^2)$.

120. The beginner will often find it convenient to write the multiplier under the multiplicand, as in arithmetic. In writing the separate products, like terms may then be written under each other in order to be added up.

EXAMPLE. $(a^2+b^2+c^2-ab-bc-ca)(a+b+c)$.

Work:

$$
\begin{array}{llll}
a^2+b^2+c^2-ab-bc-ca & & & \\
a+b+c & & & \\
\hline
a^3+ab^2+ac^2-a^2b-abc-a^2c & & & \\
\quad -ab^2 \qquad\qquad a^2b-abc & & +b^3+bc^2-b^2c & \\
\qquad\qquad -ac^2 \qquad\qquad -abc+a^2c & & & -bc^2+b^2c+c^3 \\
\hline
a^3 \qquad\qquad\qquad -3abc & & +b^3 & +c^3
\end{array}
$$

Hence product $= a^3+b^3+c^3-3abc$.

EXERCISES.

In the following exercises arrange all the terms, both of multiplicand and multiplier, according to the powers of the leading symbol.

Multiply:

1. x^2+ax+a^2 by $x-a$.
2. $x^3+ax^2+a^2x+a^3$ by $x-a$.
3. $x^4-ax^3+a^2x^2-a^3x+a^4$ by $x+a$.
4. $1-x+x^2-x^3+x^4$ by $1+x$.
5. $1-2x+3x^2-4x^4$ by $1-x+x^2$.
6. x^3-3x^2+3x-1 by x^3-2x+1.
7. x^4-x^2+1 by x^4+x^2-1.
8. a^3+2a^2+2a+1 by a^2-2a+1.
9. a^3-2a^2+2a-1 by a^2+2a+1.
10. $\frac{r^2}{a^2}-2\frac{r}{a}+3$ by $\frac{r^2}{a^2}+2\frac{r}{a}-3$.
11. $a^4-2a^3+3a^2-2a+1$ by a^3+2a^2+2a+1.
12. $(x-3)(x-1)(x+1)(x+3)$.
13. $n(n-1)(n-2)(n-3)$.
14. $(x+a)(x^2+a^2)(x-a)$.
15. $(a^2+a+1)(a^2-a+1)(a^4-a^2+1)$.
16. $a^2+4ax+4x^2$ by $a^2-4ax+4x^2$.
17. $2x^3+4x^2+8x+16$ by $3x-6$.
18. $a^2+b^2+c^2+ab+bc-ca$ by $a-b+c$.

Special Forms of Multiplication.

121. *Square of a Binomial.* To find the square of a binomial, as $a + b$. We multiply $a + b$ by $a + b$.

$$\begin{aligned} a(a + b) &= a^2 + ab \\ b(a + b) &= ab + b^2 \\ \hline (a + b)(a + b) &= a^2 + 2ab + b^2 \end{aligned}$$

Hence $$(a + b)^2 = a^2 + 2ab + b^2. \qquad (1)$$

Def. When a monomial expression is reduced to the algebraic sum of several terms, it is said to be developed.

EXERCISES.

Prove, by computing both members, that

$$(2 + 3)^2 = 2^2 + 2 \,.\, 2 \,.\, 3 + 3^2.$$
$$(4 + 3)^2 = 4^2 + 2 \,.\, 4 \,.\, 3 + 3^2.$$
$$(2 + 5)^2 = 2^2 + 2 \,.\, 2 \,.\, 5 + 5^2.$$

Write, on sight, the developed values of the following expressions:

1. $(m + n)^2$.	2. $(m + 2n)^2$.	3. $(2m + n)^2$.
4. $(ax + by)^2$.	5. $(ax + 2by)^2$.	6. $(2ax + by)^2$.
7. $(a^2 + 1)^2$.	8. $(ab + 1)^2$.	9. $(2ab + 1)^2$.
10. $(b^2 + b)^2$.	11. $(b^2 + 2b)^2$.	12. $(2b^2 + b)^2$.
13. $(m^2 + n^2)^2$.	14. $(m^2 + an)^2$.	15. $(bm^2 + abn)^2$.
16. $(b + 2)^2$.	17. $(b + 3)^2$.	18. $(b + 4)^2$.
19. $\left(\frac{x}{a} + \frac{y}{a}\right)^2$.	20. $\left(\frac{ax}{3} + \frac{by}{2}\right)^2$.	21. $\left(\frac{a}{a+b} + \frac{b}{a+b}\right)^2$.
22. $\left(a + \frac{1}{a}\right)^2$.	23. $\left(ab + \frac{n}{ab}\right)^2$.	24. $\left(\frac{a}{b} + \frac{b}{a}\right)^2$.

122. We find, in the same way,

$$(a - b)^2 = a^2 - 2ab + b^2. \qquad (2)$$

This and the preceding form may be expressed in words thus:

THEOREM. *The square of a binomial is equal to the sum of the squares of its two terms plus or minus twice their product.*

EXERCISES.

Show that the preceding formula gives correct values of $(5-2)^2$, $(5-1)^2$, and $(9-4)^2$.

Develop:

1. $(m-n)^2$.	2. $(m-2n)^2$.	3. $(2m-n)^2$.
4. $(a-1)^2$.	5. $(a-2)^2$.	6. $(a-3)^2$.
7. $(ab-x)^2$.	8. $(ab-2x)^2$.	9. $(2ab-x)^2$.
10. $(ax-by)^2$.	11. $(ax-2by)^2$.	12. $(2ax-by)^2$.
13. $\left(\frac{m}{2}-\frac{n}{3}\right)^2$.	14. $\left(\frac{x}{a}-\frac{y}{2b}\right)^2$.	15. $\left(\frac{x}{2a}-\frac{y}{b}\right)^2$.

123. *Product of the Sum and Difference.* Let us find the product of $a + b$ by $a - b$.

$$a(a+b) = a^2 + ab$$
$$-b(a+b) = \quad - ab - b^2$$

Adding, $\qquad (a+b)(a-b) = a^2 \qquad - b^2. \qquad (3)$

That is:

THEOREM. *The product of the sum and difference of two numbers is equal to the difference of their squares.*

REMARK. The three preceding forms should be carefully memorized by the student, owing to their constant occurrence.

EXERCISES.

Form the following products on sight:

1. $(m-n)(m+n)$.	2. $(m-2n)(m+2n)$.
3. $(2m+n)(2m-n)$.	4. $(ab+c)(ab-c)$.
5. $(ab+2c)(ab-2c)$.	6. $(2ab+c)(2ab-c)$.
7. $(ax+1)(ax-1)$.	8. $(ax+2)(ax-2)$.
9. $(ax+5)(ax-5)$.	10. $(ax+by)(ax-by)$.
11. $(a^2x-b^2y)(a^2x+b^2y)$.	12. $(a^2x^2-b^2y^2)(a^2x^2+b^2y^2)$.
13. $\left(\frac{x}{a}-\frac{y}{u}\right)\left(\frac{x}{a}+\frac{y}{u}\right)$.	14. $\left(\frac{x^2}{2a}-\frac{y^2}{2b}\right)\left(\frac{x^2}{2a}+\frac{y^2}{2b}\right)$.
15. $\left(a^2+\frac{1}{2}\right)\left(a^2-\frac{1}{2}\right)$.	16. $(a+b+c)(a+b-c)$.

Ans. (16). $(a+b)^2 - c^2 = a^2 + 2ab + b^2 - c^2$.

17. $(x+y+z)(x+y-z)$.	18. $(1+n+n^2)(1+n-n^2)$.
19. $(1+a+2b)(1+a-2b)$.	20. $(a+2b+2c)(a+2b-2c)$.
21. $(a+b+ab)(a+b-ab)$.	22. $(3+4-5)(3+4+5)$.

124. Because the product of two negative factors is positive, it follows that the square of a negative quantity is positive.

EXAMPLES. $(-a)^2 = a^2 = (+a)^2.$

$$(b-a)^2 = a^2 - 2ab + b^2 = (a-b)^2.$$

Hence

The expression $a^2 - 2ab + b^2$ is the square both of $a - b$ and of $b - a$.

125. We have $\quad -a \times a = -a^2.$

Hence

The product of equal factors with opposite signs is a negative square.

EXAMPLE. $\quad -(a-b)(a-b) = -a^2 + 2ab - b^2,$

which is the negative of (2). Because $-(a-b) = b-a$, this equation may be written in the form

$$(b-a)(a-b) = -a^2 + 2ab - b^2,$$

which is readily obtained by direct multiplication.

EXERCISES.

Form the products:

1. $(x-y)(y-x)$.
2. $(x+y)(y-x)$.
3. $-a^2 \times a^2$.
4. $(a-2b)(2b-a)$.
5. $\left(\frac{a}{2}-1\right)\left(1-\frac{a}{2}\right)$.
6. $(ax-by)(by-ax)$.

126. *Squares of Trinomials.* Let us form the square of

$$a + b + c.$$

$$\begin{array}{rcl} a(a+b+c) & = & a^2 + ab + ac \\ b(a+b+c) & = & b^2 + ab + bc \\ c(a+b+c) & = & c^2 + ac + bc \\ \hline \end{array}$$

$$\text{Sum} = (a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2ac + 2bc.$$

Hence

THEOREM. *The square of a trinomial is equal to the sum of the squares of its terms plus twice the product of each pair of terms taken two and two.*

This theorem applies to a polynomial of any number of terms.

EXERCISES

Develop the expressions:

1. $(a-b-c)^2$. Ans. $a^2+b^2+c^2-2ab-2ac+2bc$.

2. $(a+b-c)^2$.

3. $(x-2a+b)^2$.

4. $(a^2-ab+b^2)^2$.

5. $(a^2-2ab+b^2)^2$.

6. $(m^2+n^2-3p)^2$

7. $(1+e+e^2)^2$.

8. $(x^2-x+1)^2$.

9. $(2x^2-3x+4)^2$.

10. $(ab+bc+ca)^2$.

11. $(a^2b+b^2c+c^2a)^2$.

12. $\left(1+\frac{m}{n}+\frac{n}{m}\right)^2$.

13. $\left(\frac{m}{n}-2-2\frac{n}{m}\right)^2$.

14. $\left(a-1+\frac{1}{a}\right)^2$.

15. $(1+x+x^2+x^3)^2$.

SECTION IV. DIVISION OF COMPOUND EXPRESSIONS.

Division of one Polynomial by Another.

127. CASE I. When there is only one algebraic symbol in both dividend and divisor.

RULE. 1. *Arrange both dividend and divisor according to the powers of the symbol.*

2. *Divide the first term of the dividend by the first term of the divisor, and write down the quotient.*

3. *Multiply the whole divisor by this quotient, and subtract the product from the dividend.*

4. *Divide the first term of the remainder by the first term of the divisor, and repeat the process until the divisor will no longer divide the remainder.*

REMARK. Unless we introduce fractions, the dividing process necessarily stops when the degree of the remainder is less than that of the divisor.

EXAMPLE. Let us perform the division

$$3x^4-4x^3+2x^2+3x-1 \div (x^2-x+1).$$

We first find the quotient of the highest term of the divisor x^2 into the highest term of the dividend $3x^4$, which is $3x^2$. We then multiply the whole divisor by the quotient $3x^2$, and subtract the product from the dividend. We repeat the process on the remainder, and continue doing so until the remainder has no power of x so high as the highest

term of the divisor. The work is most conveniently arranged as follows:

$$\begin{array}{lrl|l}
 & \text{Dividend.} & & \text{Divisor.} \\
 & 3x^4 - 4x^3 + 2x^2 + 3x - 1 & & x^2 - x + 1 \\
\cline{4-4}
3x^2 \times \text{divisor,} & 3x^4 - 3x^3 + 3x^2 & & 3x^2 - x - 2 \quad \text{Quotient.} \\
\cline{2-2}
\text{First remainder,} & - x^3 - x^2 + 3x - 1 & & \\
-x \times \text{divisor,} & - x^3 + x^2 - x & & \\
\cline{2-2}
\text{Second remainder,} & - 2x^2 + 4x - 1 & & \\
-2 \times \text{divisor,} & - 2x^2 + 2x - 2 & & \\
\cline{2-2}
\text{Third and last remainder,} & 2x + 1 & &
\end{array}$$

The division can be carried no farther without fractions, because x^2 will not go into x. We now apply the same rule as in arithmetic, by adding to the quotient a fraction of which the numerator is the remainder and the denominator the divisor. The result is

$$\frac{3x^4 - 4x^3 + 2x^2 + 3x - 1}{x^2 - x + 1} = 3x^2 - x - 2 + \frac{2x + 1}{x^2 - x + 1}. \qquad (a)$$

This result may now be proved by multiplying the quotient by the divisor and adding the remainder.

EXERCISES.

Execute the following divisions, and reduce the quotients to the form (a) when there is any remainder:

1. $x^3 - 2x^2 + 2x - 1 \div x - 1$.
2. $3x^4 + 6x^3 - 2x^2 - 2x \div x^2 - 2$.
3. $48x^3 - 76x^2 - 32x + 50 \div 2x - 3$.
4. $a^2 + 1 \div a + 1$.
5. $a^3 + a^2 + a + 1 \div a + 1$.
6. $a^5 + a^3 + 2a + 1 \div a^2 + 1$.
7. $a^6 - 6a^4 + 10a^2 - 5 \div a^2 - 1$.
8. $(y^3 + 12y + 16)(y^3 + 12y - 16) \div y - 2$.
9. $(y^3 - 2y + 1)(y^3 - 3y + 2) \div (y - 1)(y^2 - 2y + 1)$.
10. $y^4 - 2y^3 + 3y^2 + 4y \div y^2 - y + 1$.
11. $(y^3 + 8)(y^2 - 4)(y + 1) \div y^2 - 2y + 1$.
12. $(y^2 - 1)(y^3 + 4)(y - 4) \div y^2 + y - 2$.
13. $(a^2 + 3)(a - 3)(a^3 - 5) \div a - 1$.
14. $(a^2 - 2)(a^3 - 4)(a - 8) \div a^2 + 2a + 1$.
15. $a(a^3 + 1)(a^2 + a)(a - 1) \div a^2 - 2a + 4$.

128. *Use of Detached Coefficients.* In dividing by the preceding method, there is no need of repeating the symbol after each coefficient.

EXAMPLE. $x^4 - 4x^3 + 35x - 11 \div x^2 + 2x - 1.$

$$
\begin{array}{rrrrr|l}
x^4 & x^3 & x^2 & x^1 & & \quad x^2 \quad x \\
+1 & -4 & 0 & +35 & -11 & +1+2-1 \\
\cline{6-6}
+1 & +2 & -1 & & & +1x^2 - 6x + 13 \\
\cline{1-4}
 & -6 & +1 & +35 & & \\
 & -6 & -12 & +6 & & \\
\cline{2-5}
 & & +13 & +39 & -11 & \\
 & & +13 & +26 & -13 & \\
\cline{3-5}
 & & & +3 & +2 &
\end{array}
$$

Here the power of x which is written above each column of coefficients is to be understood as if repeated after each coefficient below it.

Result: Quotient, $x^2 - 6x + 13$.
Remainder, $3x + 2$.

EXERCISES.

1. $a^3 - 2a^2 + 4a - 8 \div a - 2.$
2. $2a^4 + 6a^2 + 12a - 3 \div a^3 - 2a + 2.$
3. $2a^4 - 2a^3 + 6a - 6 \div a^2 - 1.$
4. $3x^5 + 6x^4 - 14x^3 - x^2 - 11x + 15 \div x^2 + 2x - 5.$
5. $x^4 - 2x^3 + 4x^2 + 2x - 5 \div x^2 - 2x + 5.$

129. CASE II. When there are several algebraic symbols in the dividend.

RULE. *Arrange the terms of the dividend and divisor according to the powers of some one symbol, preferring that symbol which has the greatest number of powers.*

Then proceed as in Case I.

EXAMPLE 1. Divide $x^3 + 3ax^2 + 3a^2x + a^3$ by $x + a$.

OPERATION.

$$
\begin{array}{llll|l}
x^3 + 3ax^2 & + 3a^2x & + a^3 & & x + a \\
\cline{5-5}
x^3 + ax^2 & & & & x^2 + 2ax + a^2 \\
\cline{1-1}
\quad 2ax^2 & + 3a^2x & & & \\
\quad 2ax^2 & + 2a^2x & & & \\
\cline{1-2}
 & \quad a^2x & + a^3 & & \\
 & \quad a^2x & + a^3 & & \\
\cline{2-3}
 & \quad 0 & \quad 0 & &
\end{array}
$$

Ex. 2. Divide $x^3 - ax^2 + a(b+c)x - abc - bx^2 - cx^2 - bcx$ by $x - a$.

Arranging according to § 116, we have the dividend as follows:

$$
\begin{array}{l|l}
x^3 - (a+b+c)x^2 + (ab+bc+ca)x - abc & x - a \\
\cline{2-2}
x^3 - \qquad\quad ax^2 & x^2 - (b+c)x + bc \\
\cline{1-1}
\quad - (b+c)x^2 + (ab+bc+ca)x & \\
\quad - (b+c)x^2 + \qquad (xb+ac)x & \\
\cline{1-1}
\qquad\qquad\qquad\qquad bcx - abc & \\
\qquad\qquad\qquad\qquad bcx - abc & \\
\cline{1-1}
\qquad\qquad\qquad\qquad\ \ 0 \qquad 0 &
\end{array}
$$

EXERCISES.

Divide the following binomials by $x + c$ and by $x - c$:

1. $x^2 + c^2$.	2. $x^3 + c^3$.	3. $x^4 + c^4$.
4. $x^5 + c^5$.	5. $x^6 + c^6$.	6. $x^7 + c^7$.
7. $x^2 - c^2$.	8. $x^3 - c^3$.	9. $x^4 - c^4$.
10. $x^5 - c^5$.	11. $x^6 - c^6$.	12. $x^7 - c^7$.

13. Divide $x^3 + (b+c)x^2 + bcx + a(x^2 + bx + cx + bc)$ by $x + a$, then by $x + b$, then by $x + c$.

14. Divide $a^4 + b^4 + c^4 - 2(a^2b^2 + b^2c^2 + c^2a^2)$ by $(a+b+c)(a+b-c)(a-b+c)(a-b-c)$.

Divide by the four factors of the divisor in succession.

15. Divide $(ax+b)^2 + (a-bx)^2$ by $x^2 + 1$.

16. Divide $x^4 + 64a^4$ by $x^2 - 4ax + 8a^2$.

17. Divide $4a^3 + 3a^2b + b^3 - 3a^2 + 3ab - 1$ by $a + b - 1$.

18. Divide $x^6 + y^6 + 2x^3y^3$ by $(x+y)^2$.

19. Divide $x^6 + y^6 - 2x^3y^3$ by $(x-y)^2$.

20. Divide $(x-2y)(x^3 - 3x^2y + 5xy^2 - 3y^3)$ by $x^2 - 3xy + 2y^2$.

21. Divide $a^8 + a^4b^4 + b^8$ by $(a^2 - ab + b^2)(a^2 + ab + b^2)$.

22. Divide $(a+2b)\,a^3 - (b+2a)b^3$ by $a - b$.

23. Divide $(x^2 + a^2)(x^2 + ax + a^2)$ by $x^4 + a^2x^2 + a^4$.

Division into Prime Factors.

In the following exercises we apply the definitions and processes of §§ 50 and 51 to compound expressions.

130. *Difference of Two Squares.* By § 123 the difference of two squares is equal to the product of the sum and difference of their square roots, which sum and difference are therefore the factors.

EXERCISES.

Factor:

1. $a^2 - b^2$. Ans. $(a + b)(a - b)$.
2. $m^2 - n^2$.
3. $4m^2 - n^2$.
4. $4m^2 - 9q^2$.
5. $16p^2 - 25q^2$.
6. $x^2 - 1$.
7. $x^2 - 4$.
8. $x^6 - a^6$. Ans. $(x^3 + a^3)(x^3 - a^3)$.
9. $x^6 - 1$.
10. $9x^6 - 4$.
11. $x^4 - a^4$. Ans. $(x^2+a^2)(x^2-a^2)=(x^2+a^2)(x+a)(x-a)$.
12. $x^4 - 1$.
13. $16a^4 - b^4$.
14. $m^2x - n^2x$. Ans. $(m^2 - n^2)x = (m + n)(m - n)x$.
15. $(a^2 - b^2)y$.
16. $(m^2 - n^2)(x + y)$.
17. $(m^2 - n^2)(x^2 - y^2)$.
18. $x^3 - xy^2$. Ans. $x(x + y)(x - y)$.
19. $a^3 - 4ax^2$.
20. $9m^4 - 4m^2n^2$.
21. $a^2 - \dfrac{1}{a^2}$
22. $\dfrac{1}{a^2} - \dfrac{4}{b^2}$.

131. *Perfect Squares.* By §§ 121 and 122 a trinomial consisting of the sum of two squares *plus* or *minus* twice the product of their square roots is the square of a binomial, and may therefore be divided into two equal factors.

EXERCISES.

Factor:

1. $x^2 - 2x + 1$. Ans. $(x - 1)^2$ or $(1 - x)^2$.
2. $x^2 - 4x + 4$.
3. $a^2 + 4ax + 4x^2$.
4. $a^2 + 6a + 9$.
5. $m^4 - 2m^2x + x^2$.
6. $x^6 + 10x^3 + 25$.
7. $49k^2 + 14k + 1$.

8. $x^3 + 2ax^2 + a^2x$. Ans. $x(x+a)^2$.
9. $a^3 - 4a^2m + 4am^2$.
10. $x^2 + 2ax + a^2 - b^2$.

Ans. $(x+a)^2 - b^2 = (x+a+b)(x+a-b)$.

Note. Here we combine the methods of this and the preceding sections.

11. $x^2 - 2ax + a^2 - b^2$.
12. $m^2 - 4m - x^2 + 4$.
13. $3a^2 - 6am + 3m^2$.

Ans. $3(a^2 - 2am + m^2) = 3(a-m)^2$.

14. $3a^2 - 6ab + 3b^2$.
15. $3a^2 - 6ab + 3b^2 - 4c^2$.
16. $5m^2 + 5n^2 - 20p^2 - 10mn$.
17. $2h^2 + 8k^2 - 8hk - 18h^2k^2$.

Ans. $2(h + 2k + 3hk)(h + 2k - 3hk)$.

18. $3p^2 - 12pq + 12q^2 - 27x^2y^2$.
19. $ax^2 + 2a^2x + a^3 - 4a^3$.
20. $mp^2 + 4m^2p + 4m^3 - 16m^5p^4$.
21. $c^2 + 2 + \dfrac{1}{c^2}$. Ans. $\left(c + \dfrac{1}{c}\right)^2$.
22. $c^2 - 2 + \dfrac{1}{c^2}$.
23. $\dfrac{a^2}{c^2} - 2 + \dfrac{c^2}{a^2}$.
24. $\dfrac{m^2}{n^2} - 4 + 4\dfrac{n^2}{m^2}$.
25. $\dfrac{1}{a^2} + \dfrac{2}{a} + 1$.
26. $\dfrac{a^2}{m^2} - \dfrac{2ac}{mn} + \dfrac{c^2}{n^2}$.
27. $\dfrac{x^2}{a^2} + 4\dfrac{xy}{ac} + 4\dfrac{y^2}{c^2}$.

132. *Other Trinomials.* If we perform the multiplication

$$(x+a)(x+b),$$

we find as the result

$$x^2 + (a+b)x + ab = (x+a)(x+b).$$

We see that in the trinomial

(1) The coefficient of x is the *sum* of a and b, and

(2) The term independent of x is their *product*.

Hence *if in a trinomial of the form*

$$x^2 + px + q$$

we can find two numbers whose sum is p and whose product is q, the trinomial may be factored.

EXAMPLE. $x^2 + 5x + 6.$

Here we find by trial that 2 and 3 are such numbers that $2 + 3 = 5$ and $2 \times 3 = 6$. Hence

$$x^2 + 5x + 6 = (x + 2)(x + 3)$$

EXERCISES.

Factor:

1. $x^2 + 3x + 2.$
2. $x^2 + 4x + 3.$
3. $x^2 + 7x + 10.$
4. $c^2 + 6c + 8.$
5. $x^4 + 7x^2 + 12.$
6. $c^4 + 8c^2 + 12.$
7. $m^6 + 9m^3 + 14.$
8. $m^6 + 8m^3 + 15.$
9. $ax^2 + 7ax + 12a.$ Ans. $a(x + 4)(x + 3).$
10. $m^2x^2 + 12mx + 35.$
11. $\dfrac{m^2}{n^2} + 13\dfrac{m}{n} + 40.$
12. $\dfrac{a^2}{c^2} + 11\dfrac{a}{c} + 24.$

133. If the second term of the trinomial to be factored is negative and the third positive, both the quantities a and b must be negative.

For the product being positive, the signs must be like, and the sum being negative, one at least must be negative. The form is

$$(x - a)(x - b) = x^2 - (a + b)x + ab.$$

EXERCISES.

Factor:

1. $x^2 - 5x + 6.$ Ans. $(x - 2)(x - 3).$
2. $x^2 - 4x + 3.$
3. $x^2 - 7x + 12.$
4. $x^2 - 9x + 18.$
5. $m^4 - 5m^2 + 4.$
6. $b^4 - 6b^2 + 8.$
7. $c^4 - 9c^2 + 20.$
8. $3x^2 - 27x + 60.$
9. $5a^2 - 25a + 20.$
10. $7a^4 - 28a^2 + 21.$
11. $\dfrac{1}{a^2} - \dfrac{12}{a} + 35.$
12. $\dfrac{m^2}{n^2} - 11\dfrac{m}{n} + 24.$
13. $\dfrac{1}{a^2} - \dfrac{3}{a} + 2.$

134. If the last term of the trinomial is negative, one of the quantities a and b must be negative and the other positive, and the coefficient of the second term must be their *algebraic* sum.

Hence, in this case, we must find two numbers whose product is the third term and whose *numerical* difference is the coefficient of x.

Then prefixing to the greater number the sign of the coefficient of x, and to the lesser number the opposite sign, we have the quantities to be added to x to form the factors.

EXERCISES.

Factor:

1. $x^2 + 2x - 8$. Ans. $(x + 4)(x - 2)$.
2. $x^3 - 2x - 8$.
3. $a^2 + 2a - 3$.
4. $a^2 - 2a - 3$.
5. $m^4 - m - 9$.
6. $m^4 + m - 6$.
7. $5x^2 - 10x - 15$. Ans. $5(x - 3)(x + 1)$.
8. $5x^2 + 10x - 15$.
9. $\frac{m^2}{n^2} + 2\frac{m}{n} - 3$.

135. By combining the above forms others may be found.

For example, the factors

$$(a^2 + ab + b^2)(a^2 - ab + b^2) \qquad (1)$$

are respectively the sum and difference of the quantities

$$a^2 + b^2 \quad \text{and} \quad ab.$$

Hence the product (1) is equal to the difference of the squares of these quantities, or to

$$(a^2 + b^2)^2 - a^2b^2 = a^4 + a^2b^2 + b^4.$$

Hence the latter quantity can be factored as follows:

$$a^4 + a^2b^2 + b^4 = (a^2 + ab + b^2)(a^2 - ab + b^2).$$

EXERCISES.

Factor:

1. $m^4 + m^2n^2 + n^4$.
2. $a^4 + 4a^2b^2 + 16b^4$.
3. $a^4 + 9a^2b^2 + 81b^4$.
4. $16a^4 + 4a^2b^2 + b^4$.
5. $3x^4 + 12a^2x^2 + 48a^4$.
6. $2b^4 + 18b^2 + 162$.
7. $16 + 4a^2 + a^4$.
8. $\frac{m^4}{n^4} + c^2\frac{m^2}{n^2} + c^4$.
9. $\frac{m^4}{n^4} + 1 + \frac{n^4}{m^4}$.
10. $\frac{a^4}{c^4} + \frac{a^2}{c^2} + 1$.
11. $\frac{a^4}{c^4} + \frac{a^2m^2}{c^2n^2} + \frac{m^4}{n^4}$.
12. $\frac{h^4}{e^4} + 9\frac{h^2}{e^2} + 81$.

☞ Teachers requiring additional exercises in factoring will find them in the Appendix.

Factors of Binomials.

136. Let us multiply
$x^{n-1} + ax^{n-2} + a^2x^{n-3} + \ldots + a^{n-2}x + a^{n-1}$ by $x - a$.

OPERATION.

$$\begin{array}{l} x^{n-1} + ax^{n-2} + a^2x^{n-3} + a^3x^{n-4} + \ldots + a^{n-2}x + a^{n-1} \\ x - a \\ \hline x^n + ax^{n-1} + a^2x^{n-2} + a^3x^{n-3} + \ldots + a^{n-1}x \\ \quad - ax^{n-1} - a^2x^{n-2} - a^3x^{n-3} - \ldots - a^{n-1}x - a^n \\ \hline \end{array}$$

Prod., $x^n \quad 0 \quad 0 \quad 0 \quad 0 \quad - a^n$

The intermediate terms all cancel each other in the product, leaving only the two extreme terms.

The product of the multiplicand by $x - a$ is therefore $x^n - a^n$. Hence if we divide $x^n - a^n$ by $x - a$, the quotient will be the above expression.

Hence the binomial $x^n - a^n$ may be factored as follows:
$x^n - a^n = (x - a)(x^{n-1} + ax^{n-2} + a^2x^{n-3} + \ldots + a^{n-2}x + a^{n-1})$.

Therefore we have:

THEOREM. *The difference of equal powers of two numbers is divisible by the difference of the numbers themselves.*

Illustration. The difference between any power of 5 and the same power of 2 is divisible by $5 - 2 = 3$. For instance,

$$5^2 - 2^2 = 21 = 3.7.$$
$$5^3 - 2^3 = 117 = 3.39.$$
$$5^4 - 2^4 = 609 = 3.203.$$

EXERCISES.

Divide the following expressions by $x - a$, and show that the quotients correspond to the above form:

1. $x^2 - a^2$. 2. $x^3 - a^3$. 3. $x^4 - a^4$. 4. $x^5 - a^5$.

Factor:

5. $x^3 - 8a^3$. 6. $8m^3 - 27n^3$. 7. $x^4 - a^3x$.

137. To find the binomials of which $x + a$ are factors, let us call b the negative of a, so that

$$b = -a \quad \text{and} \quad a = -b.$$

Then $$x + a = x - b.$$

Now, by the preceding section, $x - b$ is a factor of $x^n - b^n$. Putting $-a$ in place of b, and supposing $n = 2, 3$, etc., in succession, we have

$$\begin{aligned} x - b &= x + a \quad (\text{because } b = -a). \\ x^2 - b^2 &= x^2 - a^2 \quad (\text{because } b^2 = a^2). \\ x^3 - b^3 &= x^3 + a^3 \quad (\text{because } b^3 = -a^3). \\ x^4 - b^4 &= x^4 - a^4 \quad (\text{because } b^4 = a^4). \\ &\text{etc.} \qquad\qquad \text{etc.} \end{aligned}$$

We see that when the common exponent n of x and of a is 2, 4, 6, etc., a^n is negative, and when odd it is positive. Because all the above expressions are divisible by $x - b$, that is by $x + a$, we conclude:

THEOREM. *When n is odd, the binomial $x^n + a^n$ is divisible by $x + a$.*

When n is even, $x^n - a^n$ is divisible by $x + a$.

EXERCISES.

Divide the following expressions by $x + a$, and thus find their factors:

1. $x^3 + a^3$. 2. $x^4 - a^4$. 3. $x^5 + a^5$. 4. $x^6 - a^6$.

When n is even, $x^n - a^n$ can, by § 136, be divided by $x - a$ as well as by $x + a$. Therefore both of these quantities are factors, and the binomial may be divided by their product, $x^2 - a^2$.

EXERCISES.

Divide the following by $x^2 - a^2$, and thus factor them:

1. $x^4 - a^4$. 2. $x^6 - a^6$. 3. $x^8 - a^8$. 4. $x^{10} - a^{10}$.

Lowest Common Multiple.

138. The L.C.M. of any polynomials may be found by factoring them and applying the rule of § 55.

EXAMPLE. Find the L.C.M. of

$$2a^2 - 2b^2, \quad a^2 + 2ab + b^2, \quad a^2 - 2ab + b^2, \quad 3a^4 - 3b^4.$$

Factoring, we find these four expressions to be

$$2(a + b)(a - b), \; (a + b)^2, \; (a - b)^2, \; 3(a^2 + b^2)(a + b)(a - b).$$

Applying the rule, we find the L.C.M. to be

$$2 \cdot 3(a + b)^2(a - b)^2(a^2 + b^2).$$

EXERCISES.

Find the L.C.M. of:

1. $a^2 - b^2,\ a^4 - b^4,\ 2a + 2b.$
2. $a^3 - 1,\ a^2 + a - 2,\ a - 1,\ a + 2.$
3. $2a - 1,\ 4a^2 - 1,\ 4a^2 + 1.$
4. $x^3 - x,\ x^3 - 1,\ x^2 + 1,\ x + 1.$
5. $x,\ x - a,\ x^2 - a^2,\ x + a.$
6. $x - a,\ x^2 - ax,\ x^3 - ax^2.$
7. $4y^2 - 4b^2,\ 4y^2 + 4b^2,\ 2y + 2b.$
8. $x^2 - a^2,\ x^3 - a^3.$
9. $c,\ a,\ a - c,\ a + c.$
10. $27a^3 - 8c^3,\ 9a^2 - 4c^2,\ 3a + 2c.$
11. $2a - b,\ 4a^2 - b^2,\ 4a^2 + b^2.$
12. $a^2 + 4ab + 4b^2,\ a^2 - 4ab + 4b^2,\ a^2 - 4b^2.$
13. $x(y + z),\ y(x - z),\ xyz.$
14. $m - n;\ m^2 - n^2,\ m^3 - n^3.$
15. $m - n,\ m^2 - 2mn + n^2,\ m^2 - n^2.$
16. $a + b,\ a^2 - b^2,\ a^3 + b^3.$
17. $4c^2 - 4cn + n^2,\ 4c^2 - n^2.$
18. $a + 1,\ a^2 + 1,\ a^3 - 1.$
19. $x - 4,\ x^2 - 16,\ x^3 - 8,\ x - 2.$
20. $a - b,\ b - a,\ a + b,\ b + a.$
21. $x - y,\ y^2 - x^2,\ x + y.$
22. $p - 2q,\ q - 2p.$
23. $b^2 - a^2c^2,\ ac - b,\ a^2c^2 - 2abc + b^2.$
24. $ax + ay,\ x + y,\ a.$
25. $cx + cy,\ c^2x^2 - c^2y^2,\ c^2.$
26. $2ax,\ 3a^2x^2,\ 4a^3x^2.$

SECTION V. FRACTIONS.

139. Fractions having compound expressions for the numerators or denominators may be aggregated, multiplied, divided or reduced by the methods of §§ 57 to 76. The general rule to be followed is:

1. *Observe what part of the expression constitutes the numerator, and what the denominator.*

2. *Operate on these expressions according to the rules prescribed for fractions.*

EXERCISES.

140. *Multiplication by Entire Quantities.* Execute the following multiplications:

1. $\dfrac{a+b}{x-y} \times (a-b)$. Ans. $\dfrac{a^2-b^2}{x-y}$.

2. $\dfrac{a+2b}{x-y} \times (a+2b)$.

3. $\dfrac{2x-y}{a+b} \times (2x-y)$.

4. $\dfrac{ax+by}{m+n} \times (ay-bx)$.

5. $\dfrac{1+x+2x^2}{1-x+2x^2} \times (1-2x)$.

6. $\dfrac{a^2-ab+b^2}{a^2+ab} \times (a-b)$.

7. $\dfrac{a+b}{x^2-y^2} \times (x+y)$.

Here, because $x^2 - y^2 = (x-y)(x+y)$, the multiplier is a factor of the denominator; so we operate by § 60, getting $\dfrac{a+b}{x-y}$ as the answer.

8. $\dfrac{m+n}{x^2-4y^2} \times (x-2y)$.

9. $\dfrac{1}{a^2-2ab+b^2} \times (a-b)$.

10. $\dfrac{a+x}{a^2-4ax+4x^2} \times (a^2-4x^2)$.

Here denominator and multiplier have the common factor $a-2x$, which we suppose cancelled. Multiplying by the remaining factor, we get

$$\frac{(a+x)(a+2x)}{a-2x} = \frac{a^2+3ax+2x^2}{a-2x}. \quad \text{Ans.}$$

11. $\dfrac{m+n}{m^2-2mn+n^2} \times (m^2-n^2)$.

12. $\dfrac{3a}{a^3+b^3} \times (a+b)$.

13. $\dfrac{a+2b}{a^4-b^4} \times (a^2+b^2)$.

14. $\dfrac{a-2b}{a^3-b^3} \times (a-b)$.

15. $\dfrac{c}{b^2-2bc+c^2} \times (b+c)$.

16. $\dfrac{c}{b^2+2bc+c^2} \times (b-c)$.

17. $\dfrac{a}{a+b} \times (a^2-b^2)$.

Here the denominator is a factor of the multiplier, so the product is an entire quantity, namely,

$$a \times (a-b) = a^2 - ab.$$

18. $\dfrac{m}{m-n} \times (m^2-2mn+n^2)$.

19. $\dfrac{a+b}{a-b} \times (a^2-b^2)$.

20. $\frac{1}{x^2-9y^2}\times(x^2+9y^2)$. 21. $\frac{1}{x+4y}\times(x^2-16y^2)$.

22. $\frac{x+2y}{x-2y}\times(x^3-8y^3)$. 23. $\frac{x+y}{ax-ay}\times(x-y)$.

24. $\frac{a-b}{ax+bx}\times(a^2-b^2)$. 25. $\frac{a-x}{a^2+ax}\times(a^2-x^2)$.

141. *Dividing by dividing the Numerator or multiplying the Denominator* (§§ 58, 59).

1. $\frac{a^2-b^2}{a^2+b^2}\div(a-b)$. Ans. $\frac{a+b}{a^2+b^2}$.

2. $\frac{m^2-4n^2}{x^2+y^2}\div(m+2n)$.

3. $\frac{ax+ay}{a+b}\div(x+y)$.

4. $\frac{a^3+a^2b}{a-b}\div(a+b)$.

5. $\frac{1}{x+1}\div(x+1)$. Ans. $\frac{1}{x^2+2x+1}$.

6. $\frac{1}{x+1}\div(x-1)$.

7. $\frac{1}{a-2b}\div(a+2b)$. 8. $\frac{a+b}{a-b}\div(a^2-b^2)$.

9. $\frac{1}{a-b}\div(a-2b)$. 10. $\frac{m-n}{m+n}\div(m^2-n^2)$.

11. $\frac{b-c}{b+2c}\div(2b+c)$. 12. $\frac{a^2-b}{m-2n}\div(m+n)$.

13. $\frac{bx-by}{x+y}\div(x^2-y^2)$. 14 $\frac{1}{x-y}\div(x^2-y^2)$.

15. $\frac{x+y}{x^2+xy+y^2}\div(x^2-y^2)$.

142. *Reduction to Lowest Terms* (§ 65).

1. $\frac{x-y}{x^2-y^2}$. 2. $\frac{x-a}{x^3-a^3}$.

3. $\frac{x+a}{x^3+a^3}$. 4. $\frac{x-a}{x^4-a^4}$.

5. $\frac{x^2-2xy+y^2}{x^2-y^2}$. 6. $\frac{a-b+c}{a^2-2ab+b^2-c^2}$.

7. $\dfrac{am + bm + 2mx}{a^2 + b^2 - 4x^2 + 2ab}$.

8. $\dfrac{b^2 + 6bc + 9c^2}{b^2 + 3bc}$.

9. $\dfrac{a^2x^2 - 2abxy + b^2y^2}{a^2x^2 - b^2y^2}$.

10. $\dfrac{a^4 - 2a^3b + a^2b^2}{a^4 - a^2b^2}$.

11. $\dfrac{c^4 + 4c^3x + 4c^2x^2}{m^2c^2 - 4m^2x^2}$.

12. $\dfrac{1 - x}{1 - x^5}$.

13. $\dfrac{ax + ay}{bx + by}$.

14. $\dfrac{a^4 - a^2b^2 + b^4}{a^2 - ab + b^2}$.

15. $\dfrac{x - 3}{x^2 - 5x + 6}$.

16. $\dfrac{x - 3}{x^2 - x - 6}$.

143. *Reduction to Given Denominator* (§ 67).

1. Express $a + b$ with denominator $a + b$.

Ans. $\dfrac{(a + b)^2}{a + b} = \dfrac{a^2 + 2ab + b^2}{a + b}$.

2. Express $\dfrac{a + b}{a - b}$ with denominator $a^2 - b^2$.

3. Express $\dfrac{a + x}{a + 2x}$ " " $a^2 - 4x^2$.

4. Express $\dfrac{a}{b}$ " " $bx + by$.

5. Express $\dfrac{m}{n}$ " " $mn + n^2$.

6. Express $\dfrac{m + n}{m^2 + n^2}$ " " $m^4 - n^4$.

7. Express $\dfrac{x + 1}{x - 1}$ " " $x^4 - 1$.

8. Express $\dfrac{y}{a + 2x}$ " " $4a^2x^2 - a^4$.

9. Express $\dfrac{a}{a - x}$ " " $x^2 - a^2$.

10. Express $\dfrac{a - b - c}{a + b + c}$ with den. $a^2 + 2ac - c^2 + b^2$.

11. Express $\dfrac{1}{x - 2}$ " " $x^2 - 5x + 6$.

12. Express $\dfrac{a^2 + ab + b^2}{a^2 - ab + b^2}$ " " $a^4 - a^2b^2 + b^4$.

13. Express $\dfrac{1}{m - n - x}$ " " $m^2 - 2mn + n^2 - x^2$.

144. *Reduction to L.C.D. and Aggregation.* The following algebraic sums of fractions are to be reduced to their L.C.D., and then aggregated into single fractions by the methods of §§ 68 to 72.

1. $\dfrac{a+b}{a-b}+\dfrac{a-b}{a+b}$. Ans. $\dfrac{2(a^2+b^2)}{a^2-b^2}=\dfrac{2a^2+2b^2}{a^2-b^2}$.

2. $\dfrac{a}{a+b}+\dfrac{b}{a-b}$.

3. $\dfrac{x+2y}{x-2y}-\dfrac{x-2y}{x+2y}$.

4. $\dfrac{1}{a}-\dfrac{1}{a+b}$.

5. $\dfrac{1}{a+b}-\dfrac{1}{a-b}$.

6. $\dfrac{a}{b-c}+\dfrac{b}{c-b}$.

7. $\dfrac{1}{a}+\dfrac{1}{a+b}+\dfrac{1}{a-b}$.

8. $\dfrac{a}{a-b}-\dfrac{a^2}{a^2-b^2}+\dfrac{a^4}{a^4-b^4}$.

9. $\dfrac{a+b}{a-b}-\dfrac{a-b}{a+b}-\dfrac{4b^2}{a^2-b^2}$.

10. $\dfrac{1}{x+1}-\dfrac{x+1}{x-1}$.

11. $a+\dfrac{a^2}{b-a}$.

12. $1-\dfrac{1}{1-a}-\dfrac{1}{1-a^2}$.

13. $a+b-\dfrac{a^2}{a-b}$.

14. $a+b+\dfrac{b^2}{a-b}$.

15. $x-y-\dfrac{x-2y}{x+y}$.

16. $\dfrac{3}{1-c}-\dfrac{7}{1+c}-\dfrac{4-10c}{c^2-1}$.

17. $\dfrac{1}{a-1}-\dfrac{1}{a+2}-\dfrac{3}{a^2+4a+4}$.

18. $\dfrac{x+y}{m}-\dfrac{x-y}{n}-(x+y)\left(\dfrac{1}{m}+\dfrac{1}{n}\right)$.

19. $(m+n)\left(\dfrac{m}{n}-\dfrac{n}{m}\right)-(m-n)\left(\dfrac{m}{n}+\dfrac{n}{m}\right)$.

20. $(a-b)\left(\dfrac{1}{a}-\dfrac{1}{b}\right)+(a+b)\left(\dfrac{1}{a}+\dfrac{1}{b}\right)$.

21. $(a-1)\left(\dfrac{1}{a}+\dfrac{1}{a^2}+\dfrac{1}{a^3}\right)-(a+1)\left(\dfrac{1}{a}-\dfrac{1}{a^2}+\dfrac{1}{a^3}\right)$.

22. $(a-b)\left(1+\dfrac{b^2}{a^2-b^2}\right)-\dfrac{a^2}{a+b}$.

23. $\dfrac{1}{b-c}+\dfrac{1}{c-a}+\dfrac{1}{a-b}$.

24. $1+\dfrac{x^2-4xy+4y^2}{8xy}$.

145. *Multiplication of one Fraction by Another* (§ 75).

NOTE. In these exercises, see whether common factors can be cancelled before multiplying.

1. $\dfrac{a}{x-b} \times \dfrac{x^2-b^2}{b}$.

We see that the first denominator and the second numerator have the common factor $x-b$. So we cancel it, and only multiply

$$a \times \frac{x+b}{b} = \frac{a(x+b)}{b}.$$

2. $\dfrac{a+b}{x^2-y^2} \times \dfrac{x+y}{a^2-b^2}$. Ans. $\dfrac{1}{(a-b)(x-y)}$.

3. $\dfrac{a+b-c}{a+b+c} \times \dfrac{a-b-c}{a-b+c}$.

4. $\left(\dfrac{1-b^2}{1+c}\right)\left(\dfrac{1-c^2}{1+b}\right)\left(1+\dfrac{b}{1-b}\right)$.

5. $\dfrac{a+b}{a-b}\left(a-\dfrac{a^2}{a-b}+\dfrac{b^2}{a+b}\right)$.

6. $\dfrac{x-2y}{x+6y}\left(-1+\dfrac{x}{x-2y}-\dfrac{y}{x+2y}\right)$.

7. $\dfrac{a^2+ab}{a^2+b^2} \times \dfrac{a^3-b^3}{a^2b+ab^2}$.

8. $\dfrac{a^2-2ax+x^2}{a+x} \times \dfrac{x}{a-x} \times \dfrac{a^3+x^3}{x^2}$.

9. $\dfrac{1}{a^4-b^4} \times \dfrac{a^2+b^2}{a-b} \times \dfrac{a+b}{a-b}$.

10. $\dfrac{x+y}{x-y} \times \dfrac{x^2-y^2}{x^3+y^3} \times \dfrac{1}{x+y}$.

11. $\dfrac{a^4-b^4}{a^2-2ab+b^2} \times \dfrac{a-b}{a^2+ab}$.

12. $\left(1+\dfrac{a}{x}+\dfrac{x}{a}\right)\left(1-\dfrac{a}{x}+\dfrac{x}{a}\right)$.

146. *Division by inverting the Divisor* (§ 76).

The dividend and divisor are first to be aggregated and reduced if necessary.

1. $n^2-\dfrac{1}{n^2} \div n-\dfrac{1}{n}$.

$$\frac{n^4-1}{n^2} \div \frac{n^2-1}{n} = \frac{n^4-1}{n^2} \times \frac{n}{n^2-1} = \frac{n^2+1}{n}. \text{ Ans.}$$

2. $n^2 - \frac{1}{n^2} \div \left(n + \frac{1}{n}\right).$

3. $1 + \frac{1}{x-1} \div \left(1 - \frac{1}{x+1}\right).$

4. $\frac{a}{a+b} + \frac{b}{a-b} \div \frac{a-b}{a+b}.$

5. $\frac{n+1}{n-1} - \frac{n-1}{n+1} + 4n \div \frac{2n^2}{n+1}.$

6. $\frac{ab}{a+b} - \frac{a^2}{a-b} \div \frac{a^2+b^2}{a+b}.$

7. $1 - \frac{a}{x+a} - \frac{x}{x-a} \div \frac{2x}{x-a}.$

8. $\frac{a+b}{a-b} - \frac{a-b}{a+b} \div \frac{2a}{a+b}.$

9. $\frac{1}{x-1} - \frac{1}{x+1} - \frac{2}{x^2} \div \frac{2}{x^2(x^2-1)}.$

10. $\frac{a+b}{a-b} - \frac{m-n}{m+n} \div \frac{1}{(a-b)(m+n)}.$

11. $1 + \frac{1}{c} + \frac{1}{c^2} + \frac{1}{c^3} \div \left(\frac{1}{c} + \frac{1}{c^2}\right).$

12. $1 + \frac{a}{a-x} + \frac{x^2}{a^2-x^2} \div \frac{a}{x+a}.$

147. *Reduction of Complex Fractions.*

Def. A complex fraction is one either or both of whose terms is fractional.

The minor fractions are those which enter into the terms of the complex fraction. Their terms are called *minor terms.*

PROBLEM. To reduce a complex fraction to a simple one.

RULE. *Multiply both terms by the L.C.M. of the minor denominators.*

Reason. 1. The value of the fraction remains unchanged (§ 64).

2. The minor denominators are removed.

EXAMPLE 1. Reduce $\dfrac{\frac{m}{n} + \frac{n}{m}}{\frac{m}{n} - \frac{n}{m}}$ to a simple fraction.

The minor denominators are m and n, and their L.C.M. is mn. Then:

Numerator, $\dfrac{m}{n} + \dfrac{n}{m} \times mn = m^2 + n^2$.

Denominator, $\dfrac{m}{n} - \dfrac{n}{m} \times mn = m^2 - n^2$.

Therefore the given fraction is equal to $\dfrac{m^2 + n^2}{m^2 - n^2}$.

Ex. 2. $$\frac{\frac{m}{n}}{\frac{p}{q}} = \frac{\frac{m}{n} \times nq}{\frac{p}{q} \times nq} = \frac{mq}{np}.$$

EXERCISES.

Reduce the following complex fractions to simple ones:

1. $\dfrac{1 - \frac{a}{b}}{1 + \frac{a}{b}}$.

2. $\dfrac{1 + \frac{a}{b}}{1 - \frac{a}{b}}$.

3. $\dfrac{m - \frac{m^2}{x}}{m - \frac{m^2}{y}}$.

4. $\dfrac{\frac{a}{b}}{\frac{x}{y}}$.

5. $\dfrac{\frac{a}{b}}{\frac{b}{a}}$.

6. $\dfrac{\frac{a+b}{a-b}}{\frac{a-b}{a+b}}$.

7. $\dfrac{1 + \frac{n+1}{n-1}}{\frac{n+1}{n-1} - 1}$.

8. $\dfrac{\frac{a}{a+b} + \frac{b}{a-b}}{\frac{a}{a+b} + \frac{a}{a-b}}$.

9. $1 - \dfrac{1}{1 + \frac{a}{x}}$.

10. $\dfrac{1}{1 - \frac{a}{x}} - 1$.

11. $\dfrac{\frac{a+b}{b} - \frac{b}{a+b}}{1 - \frac{a}{b+a}}$.

12. $\dfrac{1 + \frac{x}{1-x}}{\frac{1+x}{1-x}}$.

13. $\dfrac{1-\dfrac{1}{1-x}}{x-\dfrac{x}{1+x}}$.

14. $\dfrac{1}{1-\dfrac{x}{x+y}}$.

15. $\dfrac{m-n}{1+\dfrac{m-n}{m+n}}$.

16. $\dfrac{\dfrac{x+a}{x-a}-\dfrac{x-a}{x+a}}{\dfrac{x+a}{x-a}+\dfrac{x-a}{x+a}}$.

17. $\dfrac{\dfrac{a}{x}+\dfrac{b}{y}+\dfrac{c}{z}}{\dfrac{m}{x}+\dfrac{n}{y}+\dfrac{p}{z}}$.

18. $\dfrac{\dfrac{a}{a-b}-\dfrac{b}{a+b}}{\dfrac{a}{a+b}+\dfrac{b}{a-b}}$.

19. $\dfrac{1}{\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}}$.

20. $\dfrac{\dfrac{m}{n}-\dfrac{m-n}{m+n}}{\dfrac{m}{n}+\dfrac{m+n}{m-n}}$.

SECTION VI. SUBSTITUTION.

148. In algebra we often have to substitute an expression for a single letter in some other expression. This is done by making the substitution as in Chapter II., § 109, and then reducing.

EXERCISES.

Make the following substitutions:

1. In $x^2+\dfrac{1}{x^2}$ substitute $x=\dfrac{1}{a}$. Ans. $\dfrac{1}{a^2}+a^2$.

2. In $\dfrac{1}{1-x}$ substitute $x=\dfrac{1}{a}$.

3. In $\dfrac{x}{1-x}$ substitute $x=1-a$.

4. In $\dfrac{1}{1-x}$ substitute $x=\dfrac{a}{1-a}$.

5. In $1-\dfrac{1}{x}$ substitute $x=1-\dfrac{1}{a}$.

6. In $x^2+2+\dfrac{1}{x^2}$ substitute $x=\dfrac{1}{1-b}$.

7. In $ax^2 + b^2x$ substitute $x = a - b$.

8. In $\dfrac{a}{x^2} - \dfrac{x^2}{a}$ substitute $x = \dfrac{a}{b}$.

9. In $\dfrac{a^2}{x} - \dfrac{x^2}{a}$ substitute $x = a$.

10. In $\dfrac{a+x}{a-x}$ substitute $x = am$.

11. In $\dfrac{x-a}{x+a}$ substitute $x = \dfrac{a^2}{a-c}$.

12. If $x = \dfrac{1}{1-b}$ and $b = \dfrac{1}{1-c}$, find x in terms of c.

SECTION VII. THE HIGHEST COMMON DIVISOR.

The G.C.D. of Two Numbers.

149. THEOREM I. *If two numbers have a common divisor, their sum will have that same divisor.*

Proof. Let d be a common divisor;
m the quotient of one number divided by d;
n the quotient of the other number divided by d.

Then the two numbers will be

$$dm \quad \text{and} \quad dn;$$

and their sum is $d(m + n)$.

This sum is evidently divisible by d; and the quotient $m + n$ is a whole number because m and n are whole numbers; hence follows the theorem as enunciated.

THEOREM II. *If two numbers have a common divisor, their difference will have that same divisor.*

Proof. Almost the same as in the last theorem.

Cor. If one number divides another exactly, it will divide all multiples of that other exactly.

REMARK. The preceding theorems may be expressed as follows:

A common divisor of two numbers is a common divisor of their sum, difference and multiples.

REMARK. If one number is not exactly divisible by another, a remainder less than the divisor will be left over. If we put

$$D \equiv \text{the dividend;}$$
$$d \equiv \text{the divisor;}$$
$$q \equiv \text{the quotient;}$$
$$r \equiv \text{the remainder;}$$

we shall have $$D = dq + r,$$
or $$D - dq = r.$$

EXAMPLE. 7 goes into 66 9 times and 3 over. Hence this means

$$66 = 7 \,.\, 9 + 3, \quad \text{or} \quad 66 - 7 \,.\, 9 = 3.$$

150. PROBLEM. *To find the greatest common divisor of two numbers.*

Let M and N be any two numbers, and let M be the greater.

1. Divide M by N. If the remainder is zero, N will be the common divisor required, because every number divides itself. If there is a remainder, let q be the quotient and R the remainder.

Then $$M - Nq = R.$$

Let d be the common divisor required.

Because M and N are each divisible by d, $M - Nq$ must also be divisible by d (Theorem II.). Therefore

R is divisible by d.

Hence:

α. Every common divisor of M and N is also a common divisor of N and R.

Conversely, because

$$M = Nq + R,$$

β. Every common divisor of N and R is also a divisor of M, and therefore a common divisor of M and N.

Comparing α and β we see that whatever common divisors M and N may have, those same common divisors and no others have N and R.

Therefore the greatest common divisor of M and N is the same as the greatest common divisor of N and R, and we proceed with these last two numbers as we did with M and N.

2. Let R go into N q' times with the remainder R'.

Then $$N = Rq' + R',$$
or $$N - Rq' = R'.$$

Then it can be shown, as before, that d is a divisor of R', and therefore the greatest common divisor of R and R'.

3. Dividing R by R', and continuing the process, one of two results must follow. Either—

α. We at length reach a remainder 1, in which case the two numbers are prime to each other; or,

β. We have a remainder which exactly divides the preceding divisor, in which case this remainder is the common divisor required.

To clearly exhibit the process, we express the numbers M, N and the successive remainders in the following form:

$$\begin{array}{llr} M = N.q + R, & & (R < N); \\ N = R.q' + R', & & (R' < R); \\ R = R'.q'' + R'', & & (R'' < R'); \\ R' = R''.q''' + R''', & & (R''' < R''); \\ \text{etc.} \quad \text{etc.} & & \text{etc.}, \end{array}$$

until we reach a remainder equal to 1 or 0, when the series terminates.

EXAMPLE. Find the G.C.D. of 240 and 155.

240 ÷ 155 gives quotient 1 and remainder 85;
155 ÷ 85 gives quotient 1 and remainder 70;
85 ÷ 70 gives quotient 1 and remainder 15;
70 ÷ 15 gives quotient 4 and remainder 10;
15 ÷ 10 gives quotient 1 and remainder 5;
10 ÷ 5 gives quotient 2 and remainder 0.

Therefore 5 is the greatest common divisor.

EXERCISES.

Find the G.C.D. of the following pairs of numbers, arranging the work as in the above example:

1. 12 and 56.
2. 232 and 144.
3. 96 and 156.
4. 108 and 153.
5. 72 and 102.
6. 158 and 1024.

151. *Case of Three or more Numbers.* To find the G.C.D. of three or more numbers, we first find that of any two, and then the G.C.D. of this G.C.D. and the third number.

EXERCISES.

Find the G.C.D. of:

1. 12, 15, 39.
2. 98, 140, 217.
3. 270, 198, 153.

The H.C.D. of Two Polynomials.

152. The theorems of § 149 relating to two numbers apply equally to two polynomials. Hence we may find the H.C.D. of two polynomials by a process similar to that of finding the G.C.D. of two numbers.

EXAMPLE. Find the H.C.D. of

$$x^5 - 2x^3 - x^2 + x + 1$$

and

$$x^4 + x^3 - x - 1.$$

FIRST DIVISION.

$$\begin{array}{l|l}
x^5 - 2x^3 - x^2 + x + 1 & x^4 + x^3 - x - 1 \\ \cline{2-2}
x^5 + x^4 - x^2 - x & x - 1 \\ \cline{1-1}
-x^4 - 2x^3 + 2x + 1 & \\
-x^4 - x^3 + x + 1 & \\ \cline{1-1}
-x^3 + x = \text{first remainder.} &
\end{array}$$

SECOND DIVISION.

$$\begin{array}{l|l}
x^4 + x^3 - x - 1 & -x^3 + x \\ \cline{2-2}
x^4 - x^2 & -x - 1 \\ \cline{1-1}
x^3 + x^2 - x - 1 & \\
x^3 - x & \\ \cline{1-1}
x^2 - 1 = \text{second remainder.} &
\end{array}$$

THIRD DIVISION.

$$\begin{array}{l|l}
-x^3 + x & x^2 - 1 \\ \cline{2-2}
-x^3 + x & -x \\ \cline{1-1}
0 \quad 0 = \text{third remainder.} &
\end{array}$$

Thus $x^2 - 1$ is the H.C.D. sought.

EXERCISES.

Find the H.C.D. of:

1. $x^4 - 1$ and $x^3 - 1$.
2. $x^6 - a^6$ and $x^4 - a^4$.
3. $x^2 - 6x + 8$ and $4x^3 - 21x^2 + 15x + 20$.
4. $x^3 + x^2$ and $x^4 - 1$.

153. *Case of Factors found by Sight.* When a common factor of the two polynomials can be found by simple inspection, we remove this factor from both polynomials, find the H.C.D. of the quotients and multiply it by the factor.

EXAMPLE. The polynomials

$$x^4 + x \quad \text{and} \quad x^5 + x^4 + x^3 + x^2 \tag{1}$$

have the common factor x. This factor is therefore a factor of the H.C.D. sought. Now if we divide it out the polynomials will become

$$x^3 + 1 \quad \text{and} \quad x^4 + x^3 + x^2 + x. \tag{2}$$

If now we find the H.C.D. of these expressions (2) and call it D, then Dx will be the H.C.D. of the given polynomials (1). We shall find, by going through the process,

$$D = x + 1.$$

Therefore the H.C.D. of (1) is

$$x^2 + x.$$

154. *Throwing out Factors.* If we carry on the preceding process without modification, we shall commonly find that numerical fractions enter into the remainders. These may be avoided by applying the following principle:

If a divisor contains any factor prime to the dividend, it may be rejected before dividing.

The reason of this is that we are seeking, in the final result, only for the product of all those factors which are common to *both* divisor and dividend. Therefore a factor contained in one but not in the other is not a factor of the H.C.D. sought, and hence may be rejected.

For a similar reason, *we may multiply any dividend by any factor prime to the divisor.*

EXAMPLE. Find the H.C.D. of

$$x^5 - 4x^4 + 12x^3 + 4x^2 - 13x$$

and

$$x^4 - 2x^3 + 4x^2 + 2x - 5.$$

FIRST DIVISION.

$$\begin{array}{rrrrrr|l}
x^5 - 4x^4 & + 12x^3 & + 4x^2 & - 13x & & & x^4 - 2x^3 + 4x^2 + 2x - 5 \\
\cline{7-7}
x^5 - 2x^4 & + 4x^3 & + 2x^2 & - 5x & & & x - 2 \\
\cline{1-4}
- 2x^4 & + 8x^3 & + 2x^2 & - 8x & & & \\
- 2x^4 & + 4x^3 & - 8x^2 & - 4x & + 10 & & \\
\cline{1-5}
& 4x^3 & + 10x^2 & - 4x & - 10 & & = \text{first remainder.}
\end{array}$$

This remainder contains the factor 2, which is not contained in the dividend. So we divide by it. But then the first term of the next divisor, $2x^3$, will still not go into x^4

without a fractional quotient. So we multiply the new dividend by 2.

SECOND DIVISION.

$$\begin{array}{l|l}
2x^4 - 4x^3 + 8x^2 + 4x - 10 & 2x^3 + 5x^2 - 2x - 5 \\ \cline{2-2}
\underline{2x^4 + 5x^3 - 2x^2 - 5x} & x - \frac{9}{2} \\
\quad - 9x^3 + 10x^2 + 9x - 10 & \\
\quad \underline{- 9x^3 - \frac{45}{2}x^2 + 9x + \frac{45}{2}} & \\
\qquad\qquad \frac{65}{2}x^2 \qquad - \frac{65}{2} = \text{second remainder,} &
\end{array}$$

or $\qquad \frac{65}{2}(x^2 - 1) = \text{second remainder.}$

To have avoided all fractions, we should have multiplied the dividend by 4. But we could not know this until after we had begun the division, and the failure to multiply does no harm.

We now reject the factor $\frac{65}{2}$ from the remainder, leaving $x^2 - 1$ as the next divisor.

THIRD DIVISION.

$$\begin{array}{l|l}
2x^3 + 5x^2 - 2x - 5 & x^2 - 1 \\ \cline{2-2}
\underline{2x^3 \qquad\quad - 2x\qquad} & 2x + 5 \\
\qquad 5x^2 - 5 & \\
\qquad \underline{5x^2 - 5} & \\
\qquad 0 \qquad 0 = \text{third remainder.} &
\end{array}$$

Hence $x^2 - 1$ is the H.C.D. sought.

EXERCISES.

Find the H.C.D. of:

1. $x^6 + x^3$ and $x^4 - 2x^3 + 2x^2 - 2x + 1$.
2. $2x^3 + x^2 - 5x + 2$ and $4x^3 - 4x^2 - 5x + 3$.
3. $x^3 + 1$ and $x^3 + ax^2 + ax + 1$.
4. $x^4 - 3x^3 + 2x^2 + x - 1$ and $x^3 - x^2 - 2x + 2$.
5. $2x^5 - 11x^2 - 9$ and $4x^5 + 11x^4 + 81$.
6. $6x^3 - 7ax^2 - 20a^2x$ and $3x^2 + ax - 4a^2$.
7. $12x^4 + 4x^3 + 17x^2 - 3x$ and $24x^4 - 52x^3 + 14x^2 - x$.
8. $a^4 - a^3 + 2a^2 + a + 3$ and $a^4 + 2a^3 - a - 2$.
9. $6a^4 + a^3 - a$ and $4a^3 - 6a^2 - 4a + 3$.

CHAPTER III.

EQUATIONS OF THE FIRST DEGREE.

SECTION I. EQUATIONS WITH ONE UNKNOWN QUANTITY.

155. *Def.* An equation of the first degree is one which, when cleared of fractions, contains only the first power of the unknown quantity.

All equations of the first degree may be solved by the processes of multiplication, transposition and division explained in §§ 82 to 86. These processes are embodied in the following

RULE. 1. *Clear the equation of fractions.*

2. *Transpose the terms which are multiplied by the unknown quantity to one member and those which do not contain it to the other.*

3. *Aggregate the coefficients of the unknown quantity; and*

4. *Divide both members by the coefficient of the unknown quantity.*

EXAMPLE 1. Let us take the equation

$$\frac{x-7}{2x+10}=\frac{2x-6}{4x+2}.$$

Clearing of fractions, we have

$$4x^2-26x-14=4x^2+8x-60.$$

Transposing and reducing,

$$46=34x.$$

Dividing both members by 34,

$$x=\frac{46}{34}=\frac{23}{17}.$$

This result should now be proved by computing the values of both members of the original equation when $\frac{23}{17}$ is substituted for x.

Ex. 2. $$\frac{m}{x-a} - \frac{n}{x+a} = \frac{p}{x^2-a^2}.$$

Here the L.C.M. of the denominators is $x^2 - a^2$. Multiplying each term by this factor, the equation becomes

$$m(x+a) - n(x-a) = p,$$

or $$(m-n)x + (m+n)a = p.$$

Transposing,

$$(m-n)x = p - am - an;$$

whence $$x = \frac{p - am - an}{m - n}.$$

EXERCISES.

Solve the following equations, regarding x, y or u as the unknown quantity:

1. $\frac{x}{a} + \frac{x}{b} = 1.$

2. $\frac{x}{x-a} + \frac{x}{x-b} = 2.$

3. $\frac{u+2}{u-3} = \frac{u+5}{u-1}.$

4. $\frac{u-a}{u+a} = \frac{u}{u+3a}.$

5. $\frac{4}{u+2} + \frac{7}{u+3} = \frac{37}{u^2+5u+6}.$

6. $\frac{5y-1}{7} + \frac{9y-5}{11} = \frac{9y-7}{5}.$

7. $(x-2)(x-5) = (x-8)x.$

8. $\frac{7y}{y-1} - \frac{2y}{2-y} = 0.$

9. $\frac{2}{2y-5} + \frac{1}{y-3} = \frac{6}{3y-1}.$

NOTE. When common factors appear, divide them out.

10. $\frac{1}{x-1} - \frac{1}{x-2} = \frac{1}{x-3} - \frac{1}{x-4}.$

11. $\frac{x-a}{x+b} = \frac{x-b}{x+c}.$

12. $\frac{a+b}{x-c} = \frac{a}{x-a} + \frac{b}{x-b}.$

13. $\frac{x-a}{b} + \frac{x-b}{c} + \frac{x-c}{a} = \frac{x-(a+b+c)}{abc}.$

14. $ax + b = \frac{x}{a} + \frac{1}{b}.$

15. $\frac{m(x+a)}{x+b} + \frac{n(x+b)}{x+a} = m + n.$

16. $(x-a)^2 + (x-b)^2 + (x-c)^2 = 3(x-a)(x-b)(x-c).$

17. $\dfrac{m-n}{u+m} - \dfrac{m+n}{u+n} = 0.$

18. $\dfrac{1}{x} = a.$

19. $\dfrac{1}{x} = \dfrac{1}{b}.$

20. $\dfrac{1}{x} = \dfrac{a}{b}.$

21. $\dfrac{a}{x-a} = \dfrac{b}{x+b}.$

22. $\dfrac{a}{x} + \dfrac{b}{x-a} = \dfrac{a+b}{x-b}.$

23. $\dfrac{mx}{x-m} + \dfrac{nx}{x-n} = m+n.$

24. $(x-a)^2 - (x-b)^2 = c^2.$

25. $\dfrac{m}{ax-b} - \dfrac{n}{ax+b} = 0.$

Find the value of $\dfrac{1}{x}$ from each of the following equations without clearing of fractions:

26. $\dfrac{2}{x} = 4.$ Ans. $\dfrac{1}{x} = 2.$

27. $\dfrac{3}{x} = 15.$

28. $\dfrac{m}{x} = am.$

29. $\dfrac{m+1}{x} = m.$

30. $\dfrac{m+n}{x} = m-n.$

31. $\dfrac{m}{x} + \dfrac{n}{x} = m^2 - n^2.$

32. $\dfrac{m}{2x} + \dfrac{n}{2x} = a.$

33. $\dfrac{m}{ax} + \dfrac{n}{bx} = a-b.$

Find $\dfrac{1}{z}$ and z from:

34. $\dfrac{m}{z} - \dfrac{1}{a} = \dfrac{1}{b}.$

35. $\dfrac{m+n}{z} = \dfrac{m+n}{m-n}.$

36. $\dfrac{m}{nz} = \dfrac{a}{b}.$

37. $\dfrac{m+n}{mz-nz} = \dfrac{1}{m-n}.$

38. $\dfrac{a-c}{(a+c)z} = \dfrac{a}{c}.$

39. $\dfrac{a}{z} + \dfrac{b}{2z} + \dfrac{c}{3z} = k.$

In the following equations find the value of each symbol in terms of the others:

40. $2x - 3a = 5b.$

Ans. $a = \dfrac{2x-5b}{3};$ $\quad b = \dfrac{2x-3a}{5};$ $\quad x = \dfrac{3a+5b}{2}.$

41. $5a - 4b = 2u.$

42. $7a - 14b = 21y.$

43. $ax = by.$

44. $a(x-y) = b(x+y).$

45. $\dfrac{1}{b-c} = \dfrac{2}{2a-c}.$

46. $\dfrac{ax}{by} = 2.$

SECTION II. EQUATIONS OF THE FIRST DEGREE WITH TWO UNKNOWN QUANTITIES.

156. *Def.* An equation of the first degree with two unknown quantities is one which admits of being reduced to the form

$$ax + by = c,$$

in which x and y are the unknown quantities and a, b and c represent any numbers or algebraic expressions which do not contain either of the unknown quantities.

Def. A set of several equations, each containing the same unknown quantities, is called a **system of simultaneous equations.**

157. To solve two or more simultaneous equations, it is necessary to combine them in such a way as to form one equation containing only *one* unknown quantity.

Def. **Elimination** is the process of combining equations so that one or more of the unknown quantities shall disappear.

The term "elimination" is used because the unknown quantities which disappear are *eliminated.*

There are three methods of eliminating an unknown quantity from two simultaneous equations.

Elimination by Comparison.

158. RULE. *Solve each of the equations with respect to one of the unknown quantities and put the two values of the unknown quantity thus obtained equal to each other.*

This will give a new equation with only one unknown quantity, of which the value can be found from the equation.

The value of the other unknown quantity is then found by substitution.

EXAMPLE. Solve the following set of equations:

$$\left.\begin{aligned} x + y &= 28, \\ 3x - 2y &= 29. \end{aligned}\right\}$$

From the first equation we find

$$x = 28 - y,$$

and from the second $$x = \frac{29 + 2y}{3};$$

from which we have $$28 - y = \frac{29 + 2y}{3},$$

whence $$y = 11.$$

Substituting this value in the first equation in x, it becomes

$$x = 28 - 11 = 17.$$

If we substitute it in the second, it becomes

$$x = \frac{29 + 22}{3} = \frac{51}{3} = 17,$$

the same value, thus proving the correctness of the work.

EXERCISES.

Find the values of x and y from the following equations:

1. $x + 2y = 17;\qquad x - 3y = 7.$
2. $2x - y = 8;\qquad 3x + 2y = 40.$
3. $x - 2y = 5;\qquad 3y - x = 8.$
4. $2x - 3y = 7;\qquad 2x + y = 35.$
5. $2x - 3y = m;\qquad 2x + y = n.$
6. $\frac{7}{x} = \frac{2}{y};\qquad x + y = 27.$
7. $\frac{9}{x-3} = \frac{5}{y+3};\qquad x + y = 42.$
8. $\frac{3}{y+2} = \frac{5}{x+4};\qquad 2y + x = 44.$
9. $\frac{2}{8-y} = \frac{1}{4-x};\qquad 2x + y = 30.$

Elimination by Substitution.

159. RULE. *Find the value of one of the unknown quantities in terms of the other from one equation, and substitute this value in the other equation. The latter will then have but one unknown quantity.*

EXAMPLE. We take the same example as in the preceding method, namely,

$$\begin{aligned} x + y &= 28, \\ 3x - 2y &= 29. \end{aligned}$$

From the first equation we have

$$x = 28 - y.$$

Substituting this value in the second equation, it becomes

$$84 - 3y - 2y = 29,$$

from which we obtain as before

$$y = \frac{84 - 29}{5} = 11.$$

This method may be applied to any pair of equations in four ways:

1. Find x from the first equation and substitute its value in the second.

2. Find x from the second equation and substitute its value in the first.

3. Find y from the first equation and substitute its value in the second.

4. Find y from the second equation and substitute its value in the first.

EXERCISES.

Solve the following equations in four ways:

1. $x + 2y = 18;\quad 2x - y = 6.$
2. $x - y = 1;\quad 2x + 3y = 14.$
3. $x + 2y = m;\quad 2x + y = n.$

Elimination by Addition or Subtraction.

160. Rule. *Multiply each equation by such a factor that the coefficients of one of the unknown quantities shall become numerically equal in the two equations.*

Then by adding or subtracting the equations we shall have an equation with but one unknown quantity.

Remark. We may take for the factor of each equation the coefficient of the unknown quantity to be eliminated in the other equation, unless we see that simpler multipliers will answer the purpose.

Example. Taking again the same equations as before,

$$x + y = 28,$$
$$3x - 2y = 29,$$

we multiply the first equation by 3, obtaining

$$3x + 3y = 84.$$

The coefficient of x is now the same as in the second given equation. Subtracting the second, we have

$$5y = 55,$$

whence $$y = 11.$$

Again, if we multiply the first equation by 2 and add it to the second, we have

$$5x = 85,$$

whence $$x = 17.$$

REMARK. We always obtain the same result, whatever method of elimination we use. But, as a general rule, the method of addition or subtraction is the simplest and most elegant. Very often little or no multiplication is necessary. Take the following case, for instance:

161. PROBLEM OF THE SUM AND DIFFERENCE. *The sum and difference of two numbers being given, to find the numbers.*

Let the numbers be x and y.

Let s be their sum and d their difference.

Then, by the conditions of the problem,

$$x + y = s,$$
$$x - y = d.$$

Adding the two equations, we have

$$2x = s + d.$$

Subtracting the second from the first,

$$2y = s - d.$$

Dividing these equations by 2,

$$x = \frac{s + d}{2} = \frac{s}{2} + \frac{d}{2},$$

$$y = \frac{s - d}{2} = \frac{s}{2} - \frac{d}{2}.$$

We may therefore state these conclusions in the form of the following

THEOREM. *Half the sum of any two numbers* PLUS *half their difference is equal to the greater number;* and

Half the sum of any two numbers MINUS *half their difference is equal to the lesser number.*

EXERCISES.

Solve the following pairs of equations:

1. $x + 2y = 36; \qquad x - 2y = 24.$
2. $2x + y = 8; \qquad 2x - y = 8.$
3. $3x - 5y = 17; \qquad 3x + 5y = 37.$
4. $x + 2y = 20; \qquad 2x + y = 25.$
5. $3x - 4y = c; \qquad 2x + 7y = e.$
6. $ax + by = m; \qquad ax - by = n.$
7. $ax + by = c; \qquad mx + ny = p.$
8. $ax + by = p; \qquad mx - ny = q.$
9. $\dfrac{x-5}{2} = \dfrac{y+7}{3}; \qquad \dfrac{2x-4}{2} = \dfrac{5y-8}{7}.$
10. $ax + by = c; \qquad a^2x + b^2y = h.$
11. $\dfrac{x}{a} + \dfrac{y}{b} = 1; \qquad \dfrac{x}{a} - \dfrac{y}{b} = \dfrac{1}{2}.$
12. $\dfrac{x}{a+b} + \dfrac{y}{a-b} = 2; \qquad \dfrac{x-y}{4b} = 1.$
13. $\dfrac{x-y}{x+y} = m; \qquad \dfrac{x - my + a}{x + my} = a.$
14. $\dfrac{x}{y} = \dfrac{a}{b}; \qquad \dfrac{x-b}{y-a} = 3.$
15. $\dfrac{x+y}{2} - \dfrac{x-y}{3} = a; \qquad \dfrac{x+y}{2} + \dfrac{x-y}{3} = b.$

162. Sometimes we may advantageously treat expressions containing the unknown quantities as if they were single symbols, in accordance with Principle II. of the algebraic language.

EXERCISES.

1. $5(x+y) - 2(x-y) = 44; \qquad 5(x+y) + 2(x-y) = 76.$

Solution. Taking the sum and difference of the two equations as they stand, we have

$$10(x+y) = 44 + 76 = 120, \text{ whence } x + y = 12;$$
$$4(x-y) = 76 - 44 = 32, \text{ whence } x - y = 8.$$

Finally, adding and subtracting the last pair, we have

$$2x = 20,\ 2y = 4; \text{ whence } x = 10,\ y = 2.$$

2. $3(x+2y) + 2(x-2y) = 65; \quad 3(x+2y) - 2(x-2y) = 17.$
3. $2(2x-y) + 3(2x+y) = 28; \quad 4(2x-y) + 3(2x+y) = 42.$
4. $2(5x-3y) + (4x-y) = 40; \quad 2(5x-3y) - (4x-y) = 20.$

5. $\frac{1}{x}+\frac{2}{y}=\frac{5}{24}; \quad \frac{1}{x}-\frac{2}{y}=\frac{1}{24}.$

Solution. Adding and subtracting the equations as they stand, we find

$$\frac{2}{x}=\frac{6}{24}=\frac{1}{4}, \text{ whence } \frac{1}{x}=\frac{1}{8} \text{ and } x=8;$$

$$\frac{4}{y}=\frac{4}{24}=\frac{1}{6}, \text{ whence } \frac{1}{y}=\frac{1}{24} \text{ and } y=24.$$

6. $\frac{10}{x}+\frac{8}{y}=7; \quad \frac{10}{x}-\frac{8}{y}=3.$

7. $\frac{1}{x}+\frac{1}{y}=3; \quad \frac{1}{x}-\frac{1}{y}=1.$

8. $\frac{1}{3x}+\frac{2}{5y}=\frac{1}{2}; \quad \frac{1}{3x}-\frac{2}{5y}=\frac{1}{2}.$

9. $\frac{2}{x}+\frac{3}{y}=2; \quad \frac{3}{x}-\frac{2}{y}=0.$

10. $\frac{3}{x}-\frac{4}{y}=1; \quad \frac{5}{x}-\frac{2}{y}=1.$

11. $\frac{10}{x+1}+\frac{8}{y-1}=7; \quad \frac{10}{x+1}-\frac{8}{y-1}=3.$

12. $\frac{1}{3-x}+\frac{2}{5-y}=20; \quad \frac{1}{3-x}-\frac{2}{5-y}=8.$

SECTION III. EQUATIONS OF THE FIRST DEGREE WITH THREE OR MORE UNKNOWN QUANTITIES.

163. When the values of several unknown quantities are to be found, it is necessary to have as many equations as unknown quantities.

164. *Method of Elimination.* When the number of unknown quantities exceeds two, the most convenient method of elimination is generally that by addition or subtraction. The unknown quantities are to be eliminated one at a time by the following

RULE. 1. *Select an unknown quantity to be first eliminated. It is best to begin with the quantity which appears in the fewest equations or has the simplest coefficients.*

2. *Select one of the equations containing this unknown quantity as an eliminating equation.*

3. *Eliminate the quantity between this equation and each of the others in succession.*

We shall then have a second system of equations less by one in number than the original system, and containing a number of unknown quantities one less.

4. *Repeat the process on the new system of equations, and continue the repetition until only one equation with one unknown quantity is left.*

5. *Having found the value of this last unknown quantity, the values of the others can be found by successive substitution in one equation of each system.*

EXAMPLE. Solve the equations

$$\left.\begin{array}{lrcl} (1) & 4x - 3y - z + u &=& 52; \\ (2) & x - y + 2z + 2u &=& 20; \\ (3) & 2x + 2y - z - 2u &=& 3; \\ (4) & x + 2y + z + u &=& 2. \end{array}\right\} \quad (a)$$

We shall select x as the first quantity to be eliminated, and take the last equation as the eliminating one. We first multiply this equation by three such factors that the coefficient of x shall become equal to the coefficient of x in each of the other equations. These factors are 4, 1 and 2. We write the products under each of the other equations, thus:

$$\begin{array}{lrcl} (1), & 4x - 3y - z + u &=& 52, \\ (4) \times 4, & 4x + 8y + 4z + 4u &=& 8. \\ \hline (2), & x - y + 2z + 2u &=& 20, \\ (4) \times 1, & x + 2y + z + u &=& 2. \\ \hline (3), & 2x + 2y - z - 2u &=& 3, \\ (4) \times 2, & 2x + 4y + 2z + 2u &=& 4. \\ \hline \end{array}$$

By subtracting one of each pair from the other, we obtain the equations

$$\left.\begin{array}{lrcl} (1') & 11y + 5z + 3u &=& -44, \\ (2') & 3y - z - u &=& -18, \\ (3') & 2y + 3z + 4u &=& 1. \end{array}\right\} \quad (b)$$

The unknown quantity x is here eliminated, and we have three equations with only three unknown quantities. Next we may eliminate y by using the last equation as the eliminating one. We proceed as follows:

$$
\begin{array}{ll}
(1') \times 2, & 22y + 10z + 6u = -88, \\
(2') \times 11, & 22y + 33z + 44u = 11. \\
\hline
(2') \times 2, & 6y - 2z - 2u = -36, \\
(3') \times 3, & 6y + 9z + 12u = 3. \\
\hline
\end{array}
$$

Subtracting, we have

$$
\left.
\begin{array}{ll}
(1'') & 23z + 38u = 99, \\
(2'') & 11z + 14u = 39,
\end{array}
\right\} \qquad (c)
$$

a system of two equations with only two unknown quantities.

From these equations we find:

$$
\begin{array}{ll}
(1'') \times 11, & 253z + 418u = 1089, \\
(2'') \times 23, & 253z + 322u = 897. \\
\hline
 & 96u = 192,
\end{array}
$$

whence $u = 2.$

Having thus obtained the value of one unknown quantity, we find the values of the remaining ones as follows:

From $(2'')$ we have

$$11z = 39 - 14u = 39 - 28 = 11,$$

whence $z = 1.$

From $(1')$ we have

$$
\begin{aligned}
11y &= -44 - 5z - 3u \\
&= -44 - 5 - 6 = -55,
\end{aligned}
$$

whence $y = -5.$

From (1) we have

$$
\begin{aligned}
4x &= 52 + 3y + z - u \\
&= 52 - 15 + 1 - 2 = 36,
\end{aligned}
$$

whence $x = 9.$

NOTE. The student should now verify these results by substituting the values of x, y, z and u in the four original equations (a) and see whether they are all satisfied.

EXERCISES.

1. One of the best exercises for the student will be that of resolving the previous equations (a) by taking the last equation as the eliminating one, and performing the elimination in different orders; that is, begin by eliminating u, then repeat the whole process beginning with z, etc. The final results will always be the same.

2. Find the values of x, y, z and u from the equations

$$\begin{aligned} x + y + z + u &= 4a, \\ x + y - z - u &= 4b, \\ x - y + z - u &= 4c, \\ x - y - z + u &= 4d. \end{aligned}$$

REMARK. This exercise requires no multiplication, but only addition and subtraction of the different equations.

3. $$\begin{aligned} 2x - 5y + 3z &= 17, \\ 3x + 2y - z &= 31, \\ 5x + 3y - 2z &= 52. \end{aligned}$$

4. $$\begin{aligned} ax + by + cz &= A, \\ a^2x + b^2y + c^2z &= A^2, \\ a^3x + b^3y + c^3z &= A^3. \end{aligned}$$

Many of the following equations can be simplified by adding and subtracting, so as to reach a solution more expeditiously than by following the general rule:

5. $$\begin{aligned} x + y &= a, \\ x + y + z &= b, \\ x + y + z + u &= c, \\ x - y &= d. \end{aligned}$$

6. $$\begin{aligned} x + y &= a, \\ y + z &= b, \\ z + x &= c. \end{aligned}$$

7. $$\begin{aligned} x - ny &= a, \\ y - nz &= b, \\ z - nx &= c. \end{aligned}$$

8. $$\begin{aligned} \frac{1}{x} + \frac{1}{y} &= a, \\ \frac{1}{y} + \frac{1}{z} &= b, \\ \frac{1}{z} + \frac{1}{x} &= c. \end{aligned}$$

9. $$\begin{aligned} \frac{1}{x} - \frac{1}{y} &= \frac{1}{c}, \\ \frac{1}{y} - \frac{1}{z} &= \frac{1}{a}, \\ \frac{1}{z} + \frac{1}{x} &= \frac{1}{b}. \end{aligned}$$

10. $$\begin{aligned} \frac{a}{x} + \frac{b}{y} &= 1, \\ \frac{b}{y} + \frac{c}{z} &= 1, \\ \frac{c}{z} + \frac{a}{x} &= 1. \end{aligned}$$

11. $$\begin{aligned} x - 2y &= 2, \\ x - y + z &= 5, \\ x + 2u &= 7, \\ 2x - z &= 6. \end{aligned}$$

12. $$\begin{aligned} ax + by &= c, \\ cy + bz &= a, \\ az + cx &= b. \end{aligned}$$

13. $$\begin{aligned} 3u - x &= m, \\ 3x - y &= n, \\ 3y - z &= p, \\ 3z - u &= q. \end{aligned}$$

14. $$\begin{aligned} mx - y &= 0, \\ my - z &= 0, \\ mz - x &= a. \end{aligned}$$

15.
$$x + y + z = 0,$$
$$(m + n)x + (n + p)y + (p + m)z = 0,$$
$$mnx + npy + pmz = 1.$$

16. $x + y = 2x - y + 18 = 3x - y - 6.$

NOTE. Equations of this kind, which often trouble the beginner, are easily solved by equating separately different pairs of the equal members. For instance, the second and third members are

$$2x - y + 18 = 3x - y - 6.$$

By transposing the unknown quantities to the second member, and 6 to the first, we have at once

$$24 = x \quad \text{or} \quad x = 24. \tag{1}$$

The first two members alone are

$$x + y = 2x - y + 18,$$

which gives by transposition

$$2y - x = 18,$$

from which y is found by (1).

It is to be noted that in all such cases there are as many equations as signs of equality.

17. $2x - 3y - 3 = 3x - 2y - 22 = x - y + 1.$

18. $xy - x - y + 24 = xy + x + y = xy + x - 3y + 6.$

19. $3x = 5x - y = 7x - y - 4 = z.$

20. $xy + x = xy + 3x + y - 28 = xy - 3x + 44.$

21. $2x = -3y = 5(x + y + 1).$

22. $ax = by = (a - b)(x + y + 1).$

23. $xy = (x + 5)(y - 2) = (x + 9)(y - 3).$

165. *Problems leading to Equations of the First Degree.* In the solution of the following problems the student will sometimes find it convenient to use but one unknown quantity, and sometimes to use two or more. He must always state as many equations as there are unknown quantities, but the method to be followed in the solution must be left to his own ingenuity.

1. Divide $1.25 between two boys, giving A 43 cents more than B.

Suggestion. Call x A's share and y B's share.

2. Divide a line 85 feet long into two parts of which one shall be 22 feet longer than the other.

3. Half the sum of two numbers is 95 and half their difference is 55. What are the numbers?

4. The total vote for two candidates for Congress was

20,185 and the Republican majority was 1093. How many votes were cast for each party?

5. A line being divided into two parts, the whole line *plus* the greater part is 81 feet, and the whole line *plus* the lesser part 54 feet. What is the length of the line?

6. Divide \$455 between two men, so that $\frac{1}{3}$ the share of one shall be equal to $\frac{1}{2}$ the share of the other.

7. There were subscribed \$25,000 to a college fund. A subscribed \$5000 less than B and C together, and B subscribed \$2000 more than C. What did each subscribe?

8. A man has two horses, with a saddle worth \$50. The saddle and first horse are together worth as much as the second horse; the saddle and second horse are together worth twice as much as the first horse. What is the value of each?

9. A man has a buggy and two horses worth in all \$800. When the best horse is harnessed to the buggy the team is worth \$400 more than the other horse. When the poorest horse is harnessed the team is worth \$100 more than the good horse. What is the value of the buggy and of each horse?

10. Two men have together 225 acres of land worth \$15,000. A's land is worth \$50 per acre and B's worth \$100 per acre. How much land has each?

11. A sum of \$15.60 was divided among 72 children, each boy getting 25 cents and each girl 20 cents. How many were boys and how many were girls?

12. The first of two cisterns has twice as much water as the second. If 120 gallons be poured from the first into the second, the latter will then have twice as much as the first. How much has each?

13. What fraction is that which becomes equal to $\frac{1}{2}$ when its numerator is increased by 1, and to $\frac{1}{3}$ when its numerator is diminished by 1?

Suggestion. Call $\frac{x}{y}$ the fraction.

14. What fraction becomes equal to $\frac{1}{2}$ both when 1 is subtracted from its numerator and when 2 is added to its denominator?

15. A and B each had a certain sum of money. A got \$25 more and thus had twice as much as B. Then B got

$100 more and had twice as much as A. How much had each at first?

16. In a Congressional election 17,346 votes were cast, the Democrat getting 2432 more than the National, and the Republican 878 more than the Democrat. What was the vote of each?

17. A man is 7 years older than his wife, and 10 years hence his age will be double what his wife's was 10 years ago. What are their present ages?

Suggestion. If we call x the wife's present age, what is the man's present age? What will be his age 10 years hence? What were their respective ages 10 years ago?

18. A boy is now half the age of his elder brother, but in 24 years he will be $\frac{3}{4}$ his age. What are their present ages?

19. The combined ages of a man and his wife are now 62 years, and in 11 years he will be older than she will by $\frac{1}{3}$ of her age. What are their present ages?

20. Two men having engaged in gambling, A won $4 from B and then had twice as much money as B. Then B won $25 and their shares were equal. How much had each at first?

21. A sum of money being equally divided between A and B, A got $75 more than $\frac{1}{3}$ of it. What was the sum divided?

22. What is the length of that line which being divided into three equal parts, each part is 2 inches longer than one fourth of the line?

23. A sum of $520 being divided between A, B and C, B's share was $\frac{2}{3}$ of A's share and $40 more than C's share. What was the share of each?

24. Another sum being divided, A got $40 less than half, B $40 more than one fourth, and C $40 more than one sixth. What was the amount and the share of each?

25. Two men had between them 96 dollars. A paid B one fourth of his (A's) money, but B lost $6 and then had twice as much as A. How much had each at first?

26. Two cisterns being each partly full of water, $\frac{1}{4}$ of the water in the first was poured into the second, which then had 120 gallons. Then $\frac{1}{4}$ of what was left was poured from the first into the second, when the latter had twice as much as the first. How much did each contain at first?

27. At a city election Jones had a majority of 244 votes over Smith. But it was found that $\frac{1}{8}$ of Jones's votes and $\frac{1}{10}$ of Smith's votes were illegal, and on correcting this Smith had a majority of 77. What was the legal vote of each? Ans. Smith, 10,458; Jones, 10,381.

Suggestion. Take the whole numbers of votes first cast as the unknown quantities.

28. An apple-woman bought a lot of apples at 5 for 2 cents. She sold half at 2 for a cent, and half at 3 for a cent, thus gaining 10 cents. How many apples were there? Ans. 600.

29. A train ran half the distance between two cities at the rate of 30 miles an hour and half the distance at 50 miles an hour. It performed the return journey at the uniform speed of 40 miles an hour, thus gaining half an hour on its time in going. What was the distance? Ans. 300 miles.

REMARK. In all questions involving *time*, constant *velocity* and *distance* we have the fundamental relation: *Distance* = *velocity* × *time*.

30. A man bought 29 oranges for a dollar, giving 3 cents a piece for poor ones and 4 cents for good ones. How many of each kind had he?

31. A grocer bought a lot of sugar at 8 cents a pound and of coffee at 12 cents a pound, paying \$8.60 for the whole. He sold the sugar at 10 cents a pound and the coffee at 14 cents, realizing \$10.40. How much of each did he buy?

32. A huckster bought a lot of oranges at 1 cent each and lemons at 2 cents each, paying \$1.45 for the lot. 5 of the oranges and 10 of the lemons were bad, but he sold the good fruit at 2 cents each for oranges and 3 cents for lemons, realizing \$1.90. How many of each kind did he buy?

33. The sum of the ages of two brothers is now 4 times the difference, but in 6 years it will be 5 times the difference. What are their present ages?

34. For \$6 I can buy either 4 pounds of tea and 20 pounds of coffee or 2 pounds of tea and 25 of coffee. What is the price per pound of each?

35. An almoner had 3 equal sums of money to divide between 3 families. The second family had 5 more than the first and so got \$2 apiece less, and the third had 4 more than the second and got \$1 apiece less. What were the numbers of the families and the equal amounts divided?

36. If A can do a piece of work in 3 days and B in 6 days, in what time can they do it if both work together?

Suggestion. In one day A can do $\frac{1}{3}$ and B can do $\frac{1}{6}$. Hence both together can do $\frac{1}{3}+\frac{1}{6}$. But if x be the time in which both can do it, they can do $\frac{1}{x}$ of it in a day. Hence

$$\frac{1}{x}=\frac{1}{3}+\frac{1}{6}.$$

37. If one pipe can fill a cistern in 12 minutes and another in 18 minutes, in what time can they both fill it?

38. A cistern can be emptied by two faucets in 12 minutes and by one of them in 36 minutes. In what time can it be emptied by the other?

39. A cistern can be filled by a pipe in 25 minutes when the faucet is left running, and in 15 minutes when the faucet is closed. In what time will the faucet empty it?

40. Three men can together perform a piece of work in 12 days. A can do twice as much as B, and B twice as much as C. In what time could each one separately do the work?

41. A and B can together do a job of work in 8 days, B and C in 9 days, and C and A in 12 days. In what time can each one alone do it? In what time can they all do it?

Ans. A, $20\frac{4}{7}$; B, $13\frac{1}{11}$; C, $28\frac{4}{5}$; all, $6\frac{6}{23}$.

42. A train performs the journey from Washington to Chicago in a certain time at a certain speed. By going 16 miles an hour faster it gains 9 hours, and by going 8 miles an hour slower it takes 9 hours longer. What is the original time and speed and what the distance?

43. A privateer sights an enemy's ship 9 miles away, fleeing at the speed of 6 miles an hour. If the privateer chases her at the rate of 8 miles an hour, in what time and at what distance will she overtake her?

Method of Solution. Let t be the time and d the distance. Because the privateer sails 8 miles an hour, we have $d = 8t$. The ship having 9 miles less to go will, when overtaken, have sailed $(d - 9)$ miles. Therefore $d - 9 = 6t$. From these two equations we find d and t.

44. A man gets into a stage-coach driving 7 miles an hour for a pleasure-ride; but he must walk home at the rate of 3 miles an hour, and be gone only 5 hours. How far can he ride?

45. A train performed a journey in 6 hours, going one third the way at the rate of 30 miles an hour and two thirds at the rate of 40 miles an hour. What was the distance?

46. A man had two casks of wine containing together 75 gallons. He poured one fourth the contents of the first into the second, and then poured one third the increased contents of the second cask into the first. The first then contained 23 gallons more than the second. How much did each contain at first?

47. Two casks contain between them s gallons of wine, and one had d gallons more than the other. How much wine had each?

48. A board is divided into two parts. The whole board *plus* the greater part is m feet and the whole board *plus* the lesser part is n feet. What is the length of the board?

49. Divide a board l feet long so that two thirds of one part shall be equal to one half the other part.

50. A grocer mixed k pounds of tea worth p cents a pound with h pounds worth n cents a pound. How much per pound was the mixture worth?

NOTE. The solution of this question does not really require any equation.

51. A grocer has t pounds of tea worth r cents a pound, which is formed by mixing two kinds, one worth p cents a pound and the other q cents a pound. How much of each kind did he mix?

52. A boy's age is now one third that of his elder brother, but in t years it will be one half. What are their present ages?

53. Three casks contain altogether m gallons of wine. By pouring a gallons from the first into the second and then b gallons from the second into the third, the quantities in the three casks become equal. How much did each cask contain at first?

54. A man who must be back in t hours starts in a coach going m miles an hour and walks back at the rate of n miles an hour. How far can he go and get back in time?

55. If one man can do a piece of work in a days and another in b days, in what time can they both do it?

56. If A can do the work in a days, B in b days and C in c days, in what time can they all do it if work together?

57. If A and B together can do a piece of work in m days and A alone in a days, in what time can B alone do it?

58. A man bought tea and coffee, p pounds in all, for r cents, giving m cents a pound for tea and n cents a pound for coffee. How much of each did he buy?

To prove the results add the two amounts and see whether they make p pounds.

59. Divide m dollars among 3 men, giving B a dollars more than C, and A b dollars more than B.

60. If a train makes half its journey at the rate of m miles an hour and the other half at the rate of n miles an hour, what is its average speed?

Interpretation of Negative Results.

166. An answer to a problem may sometimes come out negative. This shows that the answer must be reckoned in the opposite direction from that assumed as positive in the enunciation of the problem.

Example. A man is 30 years old and his wife is 24. In how many years will he be half as old again as she is?

Solution. Let us put t for the required number of years. In t years his age will be $30 + t$ and hers will be $24 + t$.

Because the conditions of the problem require his age to be half as much again as hers, we have

$$30 + t = \frac{3}{2}(24 + t).$$

Solving this equation, we find $t = -12$.

This negative result shows that the time when the required condition was fulfilled is not in the future but in the past, when the wife was 12 and the man 18.

Had we stated the problem, How many *years ago* was his age half as much again as hers? t would have had the opposite sign all the way through and would have come out $+12$ years, because in the enunciation years past would then be regarded as positive. Therefore whether time in the future

shall be positive or negative depends on what we assume as positive in the enunciation of the problem.

Ex. 2. A man is 4 years older than his wife, and 6 times his age is equal to 7 times hers. How many years ago was 7 times his age equal to 8 times hers?

Solution. Let us put x for her present age and t for the required number of years ago. Then his age will be $x + 4$ years, and by the conditions of the problem

$$6(x + 4) = 7x.$$

But t years ago his age must have been $x + 4 - t$, and hers must have been $x - t$. By the conditions of the problem

$$7(x + 4 - t) = 8(x - t).$$

Reducing and solving these two equations, we obtain

$$x = \quad 24,$$
$$t = - \quad 4.$$

Here t comes out negative as before; but it does not mean years ago as it did in the first example, but years in the future, because in stating the problem we assumed years in the past to be positive.

167. *Problem of the Couriers.* A courier starts from his station riding 8 miles an hour. 4 hours afterwards he is followed by another riding 10 miles an hour. How long will it require for the second to overtake the first, and what will be the distance travelled?

If t be the number of hours required, the first will have travelled $t + 4$ hours and the second t hours when they come together. Because one goes 8 miles an hour and the other 10, the whole distance travelled by the one will be $8(t + 4)$, and that travelled by the other will be $10t$. Because these distances are equal, we have

$$8t + 32 = 10t.$$

Solving this equation, we have $t = 16$, whence distance $= 160$.

Now let us change the problem thus: The first courier still riding 8 miles an hour, the second starts out after him in 2 hours and travels 6 miles an hour. In what time and at what distance will they be together?

Treating the equation in the same way as before, we find the equation to be

$$8(t + 2) = 6t$$

Solving this equation, the result is

$$t = -\ 8,$$
$$d = -48.$$

These answers being negative show that there is no point on the road in the positive direction, and no time in the future when the couriers would be together.

Thus far the algebraic result agrees with common-sense; but common-sense seems to say that the couriers would never be together, whereas the algebraic answer gives a negative time and a negative distance on the road when they were together.

The explanation of the difficulty is this. Suppose S to be the point from which the couriers started, and AB the road along which they travelled from S toward B. Suppose also that the first courier started out from

S at 8 o'clock and the second at 10 o'clock. By the rule of positive and negative quantities, distances towards A are negative. Now, because algebraic quantities do not commence at 0, but extend in both the negative and positive directions, the algebraic problem does not suppose the couriers to have really commenced their journey at S, but to have come from the direction of A, so that the first one passes S, without stopping, at 8 o'clock, and the second at 10. It is plain that if the first courier is travelling the faster, he must have passed the other before reaching S; that is, the time and distance are both negative, just as the problem gives them.

The general principle here involved may be expressed thus: *In algebra, roads and journeys, like time, have no beginning and no end.*

168. In the following problems, when negative results are obtained, the pupil should show how they are to be interpreted.

1. Albany is 6 miles below Troy on the Hudson River. Let us suppose the river to flow at the rate of 4 miles an hour. How fast must a man starting from Troy row his boat through the water in order that he may reach Albany in 1 hour?

2. The same thing being supposed, how fast must he row in order to reach Albany in 2 hours? Interpret the negative result.

3. If, starting from Albany, he rows up the river at the rate of 6 miles an hour, how long will it take him to reach Troy?

4. If he rows at the rate of 3 miles an hour, how long will it take him to reach Troy? Explain the negative result.

Apply the principle laid down by the diagram in the preceding section.

MEMORANDA FOR REVIEW.

Define: Equation of the first degree; System of simultaneous equations; Elimination.

Equations of the First Degree with One Unknown Quantity.

Give rule for solution.

Two Unknown Quantities.

Elimination by $\left\{\begin{array}{l}\text{Comparison; Rule.}\\ \text{Substitution; Rule.}\\ \text{Addition and subtraction; Rule.}\end{array}\right.$

Theorem of the Sum and Difference.

Explain theorem.

Three or More Unknown Quantities.

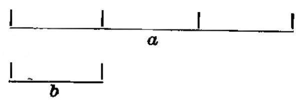
Method of elimination; Rule.

CHAPTER IV.

RATIO AND PROPORTION.

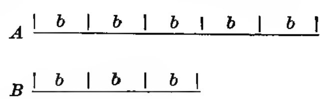
Section I. Of Ratio.

169. *Def.* When a greater quantity contains a lesser one an exact number of times, the greater is said to be a **multiple** of the lesser, and the lesser is said to **measure** the greater.



Example. If the line a contains the line b exactly three times, a is a multiple of b, and b measures a.

170. *Def.* When two quantities may each be measured by the same lesser quantity, they are said to be **commensurable**, and the lesser quantity is said to be their **common measure.**



Example. If the line A contains the measure b five times, and B contains it three times, A and B are *commensurable*, and b is their common measure.

171. *Def.* The **ratio** of two magnitudes is the number by which one must be multiplied to produce the other.

Example. In the first figure above the ratio of a to b is 3, because $b \times 3 = a$.

172. *Relation of Ratio to Quotient.* To *divide* a quantity by a number means to separate it into that *number* of equal parts. The divisor is then a number, and the quotient is a quantity of the same kind as the dividend. For instance, if we divide a line by 3, the result will be a line one third as long; if we divide weight by 4, the result will be weight one fourth as heavy.

But when we measure one line by another or one weight by another, the result is neither a line nor a weight, but a *number;* namely, the number by which one must be multiplied to produce the other. This number is the ratio. Hence another definition:

173. A ratio is the quotient of two quantities of the same kind.

A ratio is expressed by writing the divisor after the dividend with the sign : between. Thus

$$a : b = 3$$

means the ratio of a to $b = 3$, or $b \times 3$ produces a, or a divided by $b = 3$.

Hence from the equation

$$a = 3b$$

we have both $\quad a : b = 3 \quad$ and $\quad \frac{a}{3} = b.$

174. *Def.* The **antecedent** of the ratio is the quantity divided.

The **consequent** is the divisor.

Example. In the ratio $a : b$, a is the antecedent and b the consequent.

Def. Antecedent and consequent are called **terms** of the ratio.

175. When the consequent measures the antecedent, the ratio is an integer.

When the consequent and antecedent are commensurable, the ratio is a vulgar fraction.

Example. In the example of § 169, because the antecedent contains the consequent 3 times the ratio is the integer 3.

But in the example of § 170, to find the ratio $A : B$, we have to divide B into 3 parts and take 5 of these parts to make A. Hence the ratio of A to B is $\frac{5}{3}$, or

$$A : B = \frac{5}{3}.$$

The fractional ratio $\frac{5}{3}$ has for its numerator the number of times the common measure is contained in the antecedent, and for its denominator the number of times the common measure is contained in the consequent. Hence if we divide both terms into equal parts, we have the theorem:

A ratio is equal to the quotient of the number of parts in the antecedent divided by the number of equal parts in the consequent.

176. When the two terms of the ratio have no common measure they are said to be **incommensurable.**

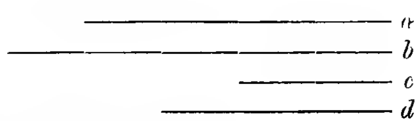
An incommensurable ratio cannot be exactly expressed as a vulgar fraction, but we can always find a vulgar fraction which shall be as near as we please to the true value of the ratio, though never exactly equal to it.

Reason. Let us divide the consequent into any number n of equal parts. Measuring the antecedent with one of these parts we shall, when the ratio is incommensurable, always have a piece of the part left over. By dropping this piece from the antecedent it will become commensurable. Now by making the number n of parts great enough, we may make each part, and therefore the piece left over, as small as we please. For example:

When $n >$ 100, piece over $<$ $\frac{1}{100}$ of consequent;
When $n >$ 1000, piece over $<$ $\frac{1}{1000}$ of consequent;
When $n >$ 1000000, piece over $<$ $\frac{1}{1000000}$ of consequent;
etc. etc.

EXERCISES.

Here are four lines of different lengths, a, b, c and d. Find as nearly as you can, by dividing up the lines, the following ratios:



1. $a : b$. Ans. 2. $a : c$. Ans.
3. $a : d$. Ans. 4. $b : c$. Ans.
5. $b : d$. Ans. 6. $c : d$. Ans.

Properties of Ratios.

177. *Def.* If we interchange the terms of a ratio, the result is called the **inverse** ratio.

That is, $B : A$ is the inverse of $A : B$.

If $$B : A = \frac{m}{n},$$

then $$B = \frac{m}{n}A,$$

and we have, by dividing by $\frac{m}{n}$,

$$A = \frac{n}{m}B,$$

or $$A : B = \frac{n}{m}.$$

Because $\frac{n}{m}$ is the reciprocal of $\frac{m}{n}$, we conclude:

THEOREM I. *The inverse ratio is the reciprocal of the direct ratio.*

178. THEOREM II. *If both terms of a ratio be multiplied by the same factor or divided by the same divisor, the ratio is not altered.*

Proof. Ratio of B to $A = B : A = \frac{B}{A}$.

If m be the factor, then

$$\text{Ratio of } mB \text{ to } mA = mB : mA = \frac{mB}{mA} = \frac{B}{A},$$

the same as the ratio of B to A.

Again, if both terms are divided by the same divisor, this operation amounts to dividing both terms of the fraction which expresses the ratio, and so leaves the value of the fraction unaltered.

179. THEOREM III. *If both terms of a ratio be increased by the same quantity, the ratio will be increased if it is less than* 1, *and diminished if it is greater than* 1; *that is, it will be brought nearer to unity.*

EXAMPLE. Let the original ratio be $2 : 5 = \frac{2}{5}$. If we repeatedly add 1 to both numerator and denominator of the fraction, we shall have the series of fractions

$$\tfrac{2}{5},\ \tfrac{3}{6},\ \tfrac{4}{7},\ \tfrac{5}{8},\ \text{etc.},$$

each of which is greater than the preceding, because

$$\begin{array}{lll} \frac{3}{6} - \frac{2}{5} = \frac{3}{30}; & \text{whence} & \frac{3}{6} > \frac{2}{5}. \\ \frac{4}{7} - \frac{3}{6} = \frac{3}{42}; & \text{whence} & \frac{4}{7} > \frac{3}{6}. \\ \frac{5}{8} - \frac{4}{7} = \frac{3}{56}; & \text{whence} & \frac{5}{8} > \frac{4}{7}. \\ \text{etc.} & & \text{etc.} \end{array}$$

General Proof. Let $a : b$ be the original ratio, and let both terms be increased by the quantity u, making the new ratio $a + u : b + u$. The new ratio *minus* the old one will be

$$\frac{a+u}{b+u} - \frac{a}{b} = \frac{(b-a)u}{b^2 + bu}.$$

If b is greater than a, this quantity wll be positive, showing that the ratio is increased by adding u. If b is less than a, the quantity will be negative, showing that the ratio is diminished by adding u.

EXERCISES.

1. What is the ratio
   (*a*) of 27 inches to 3 feet? Ans. $\frac{3}{4}$.
   (*b*) of 1122 feet to half a mile?
   (*c*) of 3 miles to 2244 yards?
2. What is the ratio
   (*a*) of $\frac{5}{3}$ of an inch to $\frac{2}{15}$ of a foot?
   (*b*) of $\frac{3}{5}$ of an inch to $\frac{5}{3}$ of an inch?
   (*c*) of $\frac{1}{5}$ of a yard to $\frac{2}{5}$ of a foot?
3. Which ratio is the greater,
   (*a*) $3 : 5$ or $5 : 8$?
   (*b*) $5 : 3$ or $8 : 5$?
   (*c*) $6 : 7$ or $7 : 8$?
   (*d*) $7 : 4$ or $14 : 8$?
4. The ratio of two numbers is $\frac{3}{5}$, and if each of them is increased by 4 their ratio will be $\frac{5}{7}$. Find the numbers.

NOTE. If we call x and y the numbers, the fact that their ratio is $\frac{3}{5}$ is expressed by the equation

$$\frac{x}{y} = \frac{3}{5}.$$

When each number is increased by 4 the ratio will be equal to $\dfrac{x+4}{y+4}$.

5. If 1 metre = 39.37 inches, what is the ratio of 10 metres to 1270 feet?

6. The ratio of two numbers is $\frac{m}{n}$, and if each of them be increased by a their ratio will become $\frac{p}{q}$. Find the numbers.

7. If the ratio of two quantities is $\frac{1}{2}$, what is the ratio of twice the antecedent to three times the consequent? What is the ratio of m times the antecedent to n times the consequent?

Section II. Proportion.

180. *Def.* **Proportion** is an equality of two or more ratios whose terms are written.

Since each ratio has two terms, a proportion must have at least four terms.

Def. The terms which enter into the two equal ratios are called **terms** of the proportion.

If $a : b$ be one of the ratios and $p : q$ the other, the proportion will be

$$a : b = p : q. \tag{1}$$

A proportion is sometimes written

$$a : b :: p : q,$$

which is read, "As a is to b, so is p to q." The first form is to be preferred, because no other sign than that of equality is necessary between the ratios; but the equation may be read, "As a is to b, so is p to q," whenever that expression is the clearer.

Def. The first and fourth terms of a proportion are called the **extremes**; the second and third are called the **means**.

Theorems of Proportion.

181. Theorem I. *In a proportion the product of the extremes is equal to the product of the means.*

Proof. Let us write the ratios in the proportion (1) in the form of fractions. It will give the equation

$$\frac{a}{b} = \frac{p}{q}. \tag{2}$$

Multiplying both members of this equation by bq, we shall have

$$aq = bp. \tag{3}$$

182. *Def.* A proportion is said to be **inverted** when antecedents and consequents are interchanged.

THEOREM II. *A proportion remains true after inversion.*

Proof. If $$a : b = p : q,$$

then $$\frac{a}{b} = \frac{p}{q};$$

and by taking the reciprocal of each member,

$$\frac{b}{a} = \frac{q}{p},$$

whence $$b : a = q : p.$$

183. THEOREM III. *If the means in a proportion be interchanged, the proportion will still be true.*

Proof. Divide the equation (3) by pq. We shall then have, instead of the proportion (1),

$$\frac{a}{p} = \frac{b}{q},$$

or $$a : p = b : q.$$

Def. The proportion in which the means are interchanged is called the **alternate** of the original proportion.

The following examples of alternate proportions should be studied, and the proof of the equations proved by calculation:

$1 : 2 = 4 : 8$;	alternate,	$1 : 4 = 2 : 8$.
$2 : 3 = 6 : 9$;	"	$2 : 6 = 3 : 9$.
$5 : 2 = 25 : 10$;	"	$5 : 25 = 2 : 10$.

184. THEOREM IV. *If in a proportion we increase or diminish each antecedent by its consequent, or each consequent by its own antecedent, the proportion will still be true.*

EXAMPLE. In the proportion

$$5 : 2 = 25 : 10$$

the antecedents are 5 and 25, the consequents 2 and 10 (§ 174). Increasing each antecedent by its own consequent, the proportion will be

$$5 + 2 : 2 = 25 + 10 : 10, \quad \text{or} \quad 7 : 2 = 35 : 10.$$

Diminishing each antecedent by its consequent, the proportion will become

$$5-2:2=25-10:10, \qquad \text{or} \qquad 3:2=15:10.$$

Increasing each consequent by its antecedent, the proportion will be

$$5:2+5=25:10+25, \qquad \text{or} \qquad 5:7=25:35.$$

These equations are all to be proved numerically.

General Proof. Let us put the proportion in the form

$$\frac{a}{b}=\frac{p}{q}. \tag{4}$$

If we add 1 to each member of this equation and reduce, it will become

$$\frac{a+b}{b}=\frac{p+q}{q}; \tag{5}$$

that is, $$a+b:b=p+q:q. \tag{6}$$

In the same way, by subtracting 1 from each side, we have

$$\frac{a-b}{b}=\frac{p-q}{q}, \tag{7}$$

or $$a-b:b=p-q:q. \tag{8}$$

If we invert the fractions in equation (4), the latter will become

$$\frac{b}{a}=\frac{q}{p}.$$

By adding or subtracting 1 from each member of this equation, and then again inverting the terms of the reduced fractions, we shall find

$$a:b+a=p:q+p; \tag{9}$$
$$a:b-a=p:q-p. \tag{10}$$

The forms (7) and (9), in which each pair of terms is added, are said to be formed by *composition*. The forms (8) and (10) are said to be formed by *division*.

185. *Composition and Division.* Taking the quotient of (5) by (7), we have

$$\frac{a+b}{a-b}=\frac{p+q}{p-q};$$

that is, $$a+b:a-b=p+q:p-q.$$

Def. This form is said to be formed from $a:b=p:q$ by composition and division.

EXERCISES.

1. From the proportion

$$2 : 7 = 6 : 21$$

form as many more proportions as you can by alternation, inversion, composition and division:

$$\begin{array}{rl}
\textit{Ans.}\ \text{Alt.} & 2 : 6 = 7 : 21. \\
\text{Inv.} & 7 : 2 = 21 : 6; \\
\text{"} & 6 : 2 = 21 : 7. \\
\text{Comp.} & 9 : 7 = 27 : 21; \\
\text{"} & 2 : 9 = 6 : 27. \\
\text{Div.} & 5 : 7 = 15 : 21; \\
\text{"} & 2 : 5 = 6 : 15. \\
\text{Comp. and div.} & 9 : 5 = 27 : 15.
\end{array}$$

2. Do the same with the proportion

$$24 : 9 = 8 : 3.$$

186. THEOREM V. *If each term of a proportion be raised to the same power, the proportion will still be true.*

Proof. If $$a : b = p : q,$$

or $$\frac{a}{b} = \frac{p}{q},$$

then, by multiplying each member by itself repeatedly, we shall have

$$\frac{a^2}{b^2} = \frac{p^2}{q^2},$$
$$\frac{a^3}{b^3} = \frac{p^3}{q^3},$$
etc. etc.

Hence, in general,

$$a^n : b^n = p^n : q^n.$$

Cor. If $$a : b = p : q,$$

then, from Th. IV., $a^n : a^n \pm b^n = p^n : p^n \pm q^n$

and $$a^n \pm b^n : b^n = p^n \pm q^n : q^n.$$

187. THEOREM VI. *When any three terms of a proportion are given, the fourth can always be found from the theorem that the product of the means is equal to that of the extremes.*

We have shown that whenever

$$a : b = p : q,$$

then $$aq = bp.$$

Considering the different terms in succession as unknown quantities, we find

$$a = \frac{bp}{q},$$

$$b = \frac{aq}{p},$$

$$p = \frac{aq}{b},$$

$$q = \frac{bp}{a}.$$

188. THEOREM VII. *If the ratio of two quantities is given, the relation between them may be expressed by an equation of the first degree.*

Proof. Let x and y be the quantities, and let their given ratio be $\frac{m}{n}$. By the definition of ratio this will mean that if we multiply y by $\frac{m}{n}$ the result will be x. That is,

$$x = \frac{m}{n}y.$$

Multiplying by n,

$$my = nx,$$

an equation of the first degree.

Cor. If we know that the ratio of two unknown quantities is $\frac{m}{n}$, we may call one of them mx and the other nx.

For $$mx : nx = \frac{m}{n}. \qquad (\S\ 178)$$

The Mean Proportional.

189. *Def.* When the middle terms of a proportion are equal, either of them is called the **mean proportional** between the extremes.

The fact that b is the mean proportional between a and c is expressed in the form

$$a : b = b : c.$$

Theorem I. then gives $b^2 = ac$.

Extracting the square root of both members, we have

$$b = \sqrt{ac}.$$

Hence

THEOREM VIII. *The mean proportional of two quantities is equal to the square root of their product.*

Def. Three quantities are said to be **in proportion** when the second is a mean proportional between the first and third, and the third is then called the **third proportional.**

EXERCISES.

What is the mean proportional between:

1. 16 and 36? 2. 2 and 8? 3. a^2 and b^2?

4. $a + b$ and $a - b$? 5. $4a$ and $9a$? 6. $2mx$ and $50mx$.

7. What is the third proportional to 4 and 6?

There are two ways of considering this question:

I. The third quantity must have the same ratio to 6 that 6 has to 4; that is, it must be $\frac{6}{4}$ of 6, which is 9.

II. Because the middle term, 6, is the square root of the product of the first and third, we must have

$$6 = \sqrt{4 \times \text{third}}, \quad \text{or} \quad \text{third} \times 4 = 36,$$

whence third $= 9$ as before.

What is the third proportional to:

8. m and mx? 9. $2m$ and $4m$? 10. m and n?

Problems in Ratio and Proportion.

1. A poulterer had 75 chickens and geese, and there were 3 chickens to every 2 geese. How many of each kind had he?

Solution. By § 188 we may call $2x$ the number of his geese, and $3x$ will then be the number of his chickens. Therefore the total number will be

$$3x + 2x = 75,$$

which gives $x = 15.$

Therefore $3x = 45 =$ number of chickens;
$2x = 30 =$ number of geese.

2. A drover had 5 sheep to every 2 cattle, and the number of his sheep exceeded that of his cattle by 102. How many had he of each?

3. A huckster had 3 apples to every 2 potatoes. Half the number of his apples added to three times the number of his potatoes made 960. How many had he of each?

4. Divide 198 into three parts proportional to the numbers 2, 3 and 4.

REMARK. We may call the parts $2x$, $3x$ and $4x$.

5. Find three numbers proportional to 1, 2 and 3 whose sum shall be 192.

6. Divide 561 into four parts proportional to 2, 3, 5 and 7.

7. Three partners, A, B and C, had 2, 3 and 6 shares respectively. On dividing a year's profits, C had $600 more than A and B together. What was the amount divided and the share of each?

8. A farmer had 3 cattle to every 2 horses, and his horses and cattle together were one third his sheep. The whole number was 160. How many had he of each?

9. Two numbers are in the ratio 3 : 5, and if 6 be taken from each the ratio will be 5 : 9. What are the numbers?

10. When a couple were married their ages were as 5 : 4. After 10 years they were as 7 : 6. What were the ages?

11. The quantity of water in two cisterns was as 3 : 5. After pouring 12 gallons from the second into the first they were as 5 : 6. How much water had each at first?

12. Find two fractions whose ratio shall be 2 : 3 and whose sum shall be unity.

13. Find two fractions of which the sum shall be unity and the ratio $x : y$.

14. What are those fractions of which the sum is $\frac{a}{b}$ and the ratio $a : b$?

Ans. $\dfrac{a^2}{ab + b^2}$ and $\dfrac{a}{a + b}$.

15. Of what two quantities is the ratio $m : n$ and the difference h?

16. Divide 184 into three parts such that the ratio of the first to the second shall be 2 : 3, and of the second to the third 5 : 7.

Suggestion. If we call x, y and z the numbers, we may state the proportions

$$x : y = 2 : 3,$$
$$y : z = 5 : 7;$$

whence, by § 181, $3x = 2y$ and $5z = 7y$.

17. Divide 232 into three parts such that the ratio of the first to the second shall be $3 : 4$, and of the first to the third $2 : 5$.

18. A poulterer had 5 chickens to every 3 ducks and 1 goose to every 4 ducks, while the entire number of all was 175. How many of each?

19. Of two trains, one went 3 miles to the other's 4, and the first went 50 miles farther in 6 hours than the other did in 3 hours. What was the speed of each?

20. At 8 o'clock a train left Washington for New York, distance 232 miles. At 9 o'clock a train left New York for Washington, going as far in 4 hours as the other did in 5. They met at the middle point of the road. What was the speed of each?

21. Of three partners, A, B and C, A had 5 shares and B had 7. C sold A $\frac{1}{3}$ of his stock and B $\frac{1}{7}$, when he had half as much as A and B together. How many shares had he at first?

22. If $x : y = 5 : 3$, find the values of the ratios

$$\begin{aligned} x &: x + y; \\ x &: x - y; \\ x + y &: x - y; \\ x - 2y &: x + 2y; \\ y - x &: y; \\ y &: 2y - x; \\ ax &: by; \\ ax + by &: ax - by. \end{aligned}$$

23. If $\frac{x + y}{x - y} = a$, find $\frac{x}{y}$.

24. If $ay = bx$, find $\frac{x + y}{x - y}$.

25. What is the ratio of the speed to the current when it takes a boat one third longer to go up stream than it does to go down?

26. When two boats were steaming in the same direction, one took 2 minutes to get a length ahead of the other. When in opposite directions, they were separated by a length in 10 seconds. What was the ratio of their respective speeds?

27. Two men ran a race to a goal and back. The swifter reached the goal in 40 seconds, and, turning back, met the slower 2 seconds later. What was the ratio of their speeds?

28. In a drove there were 2 cattle to every 7 sheep, and 5 horses to every 9 cattle. What was the ratio of the sheep to the horses?

29. It takes one boat $\frac{1}{3}$ longer to go up stream than down, and another $\frac{1}{4}$ longer. What is the ratio of their speeds?

Suggestion. First find the ratio of the speed of each to the velocity of the current.

30. One cask is filled with pure alcohol, while another, three times as large, has equal parts of alcohol and water. What is the ratio of alcohol to water if the contents of both are mixed?

31. A cask contained wine and water in the ratio 7 : 5. After adding 8 gallons of wine and 10 gallons of water the ratio was 5 : 4 and the cask was full. How much did the cask hold?

32. One ingot contains equal parts of gold and silver, and another has 2 parts of gold to 3 of silver. If I mix equal weights from the two ingots, what will be the ratio of gold to silver in the mixture?

33. A milkman had two cans, one containing milk and the other an equal quantity of water. He poured half the milk into the water, and then poured half the mixture thus formed back into the milk-can. What was then the ratio of milk to water in the milk-can?

34. The speed of two trains was as 4 to 3, and it took the swifter train 1 hour longer to make a journey of 416 miles than the slower train required for a journey of 273 miles. What was the speed of each train?

35. If a gold coin contains 9 parts of gold and 1 of silver, and a silver coin contains 6 parts of silver and 1 of copper, what proportions of gold, silver and copper will an alloy of equal weights of the two coins contain?

36. What will be the proportions when 2 parts of the gold coin, as above, are combined with 1 of the silver?

37. A goldsmith forms two alloys of equal weight, the one composed of equal parts of gold and silver, the other of two parts of gold to one of silver. If gold is 16 times as valuable as silver, what is the ratio of the values of the two alloys?

☞ Teachers requiring additional problems in ratio and proportion will find them in the Appendix.

MEMORANDA FOR REVIEW.

Ratio.

Define: Multiple; Measure; Commensurable; Common measure; Incommensurable; Ratio (in two ways); Antecedent; Consequent; Terms; Inverse ratio.

Explain and Distinguish

- Ratio as a quotient.
- The three classes of ratio $\left\{\begin{array}{l}\text{Integral,}\\ \text{Fractional,}\\ \text{Incommensurable.}\end{array}\right.$
- Relation of direct and inverse ratio.
- Result when both terms are $\left\{\begin{array}{l}\text{multiplied}\\ \text{divided}\end{array}\right.$ by the same quantity.
- Result when both terms are $\left\{\begin{array}{l}\text{increased}\\ \text{diminished}\end{array}\right.$ by the same quantity.

Proportion.

Define: Proportion; Terms; Extremes; Means; Inversion; Alternation; Composition; Division; Mean proportional; Third proportional.

Theorems of Proportion.

- Products of Extremes and Means, Theorem of; Prove.
- Inversion, Theorem of; Prove.
- Alternation, Theorem of; Prove.
- Composition; Prove.
- Division; Prove.
- Composition and division; Explain.
- Each term raised to the same power; Th.
- Express any term in terms of the three others.
- Express that two quantities have a given ratio.
- Mean proportional; Theorem of.

CHAPTER V.

POWERS AND ROOTS.

SECTION I. POWERS AND ROOTS OF MONOMIALS.

190. *Def.* The result of taking a quantity A n times as a factor is called the nth **power of** A, and, as already known, may be written either

$$AAA, \text{ etc., } n \text{ times, or } A^n.$$

Def. The exponent n is called the **index** of the power.

Def. **Involution** is the operation of finding the powers of algebraic expressions.

The operation of involution may always be indicated by the application of the proper exponent, the expression to be involved being enclosed in parentheses.

EXAMPLES. The nth power of $a + b$ is $(a + b)^n$.
The nth power of abc is $(abc)^n$.

Involution of Monomials.

191. *Involution of Products.* The nth power of the product of several factors, a, b, c, may be expressed without exponents as follows:

$$abc\ abc\ abc, \text{ etc.,}$$

each factor being repeated n times.

Here there will be altogether n a's, n b's and n c's, so that, using exponents, the whole power will be $a^nb^nc^n$

Hence $$(abc)^n = a^nb^nc^n.$$

That is,

THEOREM. *The power of a product is equal to the product of the powers of the several factors.*

EXERCISES.

Form the squares, the cubes and the nth powers of:

1. mn. 2. $2hp$. 3. $3xy$.
4. $2an$. 5. $4qr$. 6. $5pq$.

192. *Involution of Fractions.* Applying the same methods to fractions, we find that the nth power of $\frac{x}{y}$ is $\frac{x^n}{y^n}$. For

$$\left.\begin{aligned}\left(\frac{x}{y}\right)^n &= \frac{x}{y}\,\frac{x}{y}\,\frac{x}{y}, \text{ etc., } n \text{ times}\\ &= \frac{xxx, \text{ etc., } n \text{ times}}{yyy, \text{ etc., } n \text{ times}}\\ &= \frac{x^n}{y^n}.\end{aligned}\right\} \qquad (\S\ 75)$$

EXERCISES.

Form the squares, the cubes and the nth powers of:

1. $\frac{am}{bn}$. 2. $\frac{2cpq}{3xyz}$. 3. $\frac{3px}{4mqy}$. 4. $\frac{abyz}{3mnx}$.

193. *Involution of Powers.* Let it be required to raise the quantity a^m to the nth power.

Solution. The nth power of a^m is, by definition,

$$a^m \times a^m \times a^m, \text{ etc., } n \text{ times.}$$

By § 36, the exponents of a are all to be added, and as the exponent m is repeated n times, the sum

$$m + m + m + \text{etc., } n \text{ times}$$

is mn. Hence the result is a^{mn}, or, in the language of algebra,

$$(a^m)^n = a^{mn}.$$

Hence

THEOREM. *If any power of a quantity is itself to be raised to a power, the indices of the powers must be multiplied together.*

EXAMPLES.

$$(a^2)^3 = a^2a^2a^2 = a^6;$$
$$(3ab^2c^3)^4 = 81a^4b^8c^{12}.$$

EXERCISES.

Write the squares and the cubes of the following expressions:

1. ab^2. 2. $2m^3n$. 3. p^2q^3.
4. ab^2x^4. 5. $3c^2h^4x$. 6. $5g^2m^2x$.
7. $\frac{a^2}{b^2}$. 8. $\frac{a^3}{b^3}$. 9. $\frac{2h^3m}{3q^3n}$.
10. $\frac{5r^2t^n}{2m^2n^n}$. 11. $\frac{2r^nx}{3q^ny}$. 12. $\frac{m^{m+n}}{n^{p+q}}$.

Form the nth powers of:

13. p^2q^3. 14. $2m^3x$. 15. $3p^2y$.
16. $\frac{2m^p}{3a^2q}$. 17. $\frac{ab^2x^3}{cd^2y^2}$. 18. $\frac{2a^m}{3b^m}$.
19. $\frac{a^nb^n}{p^2}$. 20. $\frac{x^2y^3}{c^n}$. 21. $\frac{3c^2y^n}{4b^2x^n}$.

Reduce:

22. $(h^m)^n$. 23. $(p^m)^m$. 24. $(5m^2x^3)^2$.
25. $(2b^3)^2$. 26. $(2ab^3)^3$. 27. $\left(2\frac{m^p}{n^p}\right)^p$.

194. *Algebraic Signs of Powers.* Since the continued product of any number of positive factors is positive, all the powers of a positive quantity are positive.

By § 103, the product of an odd number of negative factors is negative, and the product of an even number is positive. Hence

THEOREM. *The even powers of negative quantities are positive, and the odd powers are negative.*

EXAMPLES. $(-a)^2 = a^2$; $(-a)^3 = -a^3$; $(-a)^4 = a^4$; etc.

EXERCISES.

Find the values of:

1. $(-3)^2$. 2. $(-3)^3$. 3. $(-3)^4$.
4. $(-a^3)^2$. 5. $(-a^2)^3$. 6. $(-a^3)^3$.
7. $(-ab^2)^2$. 8. $(-m^2x^3)^3$. 9. $(-cn^2x^3)^2$.
10. $(-1)^2$. 11. $(-1)^3$. 12. $(-1)^4$.
13. $(-h)^{2n}$. 14. $(-h)^{2n+1}$. 15. $(-h)^{2n-1}$.
16. $(-1)^{2n}$. 17. $(-1)^{2n-1}$. 18. $(-1)^{2n+1}$.
19. $\left(\frac{-m^2x}{-c^2y}\right)^2$. 20. $\left(\frac{-p^2n}{q^2s}\right)^3$. 21. $\left(\frac{2p^2u}{-k^2v}\right)^5$.

Roots of Monomials.

195. *Def.* The nth **root** of a quantity q is such a number as, being raised to the nth power, will produce q.

The number n is called the **index** of the root.

The second root is called the **square root.**

The third root is called the **cube root.**

EXAMPLES. 3 is the 4th root of 81, because

$$3 \,.\, 3 \,.\, 3 \,.\, 3 = 3^4 = 81.$$

Def. **Evolution** is the process of extracting roots.

196. *Sign of Evolution, or the Radical Sign.*

$\sqrt{\ }$, called *root*, indicates the square root of the quantity to which it is prefixed.

When any other than the square root is expressed, the index of the root is prefixed to the sign $\sqrt{\ }$.

Thus $\sqrt[3]{\ }$, $\sqrt[4]{\ }$, $\sqrt[n]{\ }$ indicate the cube, the fourth and the nth roots respectively.

The radical sign is commonly followed by a vinculum $\overline{\qquad\qquad}$ extended over the expression whose root is indicated.

EXAMPLES. In $\sqrt{a} + b + x$ the root applies to a only.

In $\sqrt{a + b} + c$ the root indicated is that of $a + b$.

In $\sqrt{a + b + c}$ the root indicated is that of $a + b + c$.

But we may use parentheses in place of the vinculum, writing $\sqrt{}(a + b) + c$, $\sqrt{}(a + b + c)$, etc.

EXERCISES.

What are the values of the following expressions?

1. $\sqrt{7 + 9}$. Ans. 4.
2. $\sqrt{20 + 5}$.
3. $\sqrt{}(3^2 + 4^2)$.
4. $\sqrt{}(17 + 8^2)$.
5. $\sqrt{6 \,.\, 15 \,.\, 10}$.
6. $\sqrt{a^2}$.
7. $\sqrt{}(a^2 + 2ab + b^2)$.
8. $\sqrt[3]{8}$.
9. $\sqrt{}(2^3 + 4^2 + 3)$.
10. $\sqrt[4]{81}$.
11. $\sqrt[4]{5^2 - 3^2}$.
12. $\sqrt[3]{14^2 - 13^2}$.

197. *Division of Exponents.* Let us extract the square root of a^6. We must find such a quantity as, being multiplied by itself, will produce a^6. It is evident that the required quantity is a^6, because, by the rule for involution (§191),

$$a^3 \times a^3 = a^6.$$

The square root of a^n will be $a^{\frac{n}{2}}$, because

$$a^{\frac{n}{2}} \times a^{\frac{n}{2}} = a^{\frac{n}{2}+\frac{n}{2}} = a^n.$$

In the same way, the cube root of a^n is $a^{\frac{n}{3}}$, because

$$a^{\frac{n}{3}} \times a^{\frac{n}{3}} \times a^{\frac{n}{3}} = a^n.$$

The following theorem will now be evident:

THEOREM. *The square root of a power may be expressed by dividing its exponent by* 2, *the cube root by dividing it by* 3, *and the nth root by dividing it by n.*

EXERCISES.

Find the cube roots of:

1. p^3. 2. p^6. 3. p^2. 4. p^{3n}.

198. Since the even powers of negative quantities are positive, it follows that an even root of a positive quantity may be either positive or negative.

This is expressed by the sign $\pm$; read, *plus or minus*.

EXAMPLES. $\sqrt{a^2} = \pm a$; $\sqrt{(a-b)^2} = a - b$ or $b - a$.

199. If the quantity of which the root is to be extracted is a product of several factors, we extract the root of each factor and take the product of these roots.

EXAMPLE. The square root of $a^4m^2p^6$ is $\pm a^2mp^3$, because $(a^2mp^3)^2 = a^4m^2p^6$, by §§191 and 193.

200. If the quantity is a fraction, we extract the roots of both members. The root of the fraction is then the quotient of the roots.

EXAMPLES. $\sqrt[3]{\frac{x^3}{y^9}} = \frac{x}{y^3}$; $\sqrt[4]{\frac{x^4}{c^6}} = \pm\frac{x}{c^2}$.

EXERCISES.

Extract the square roots of:

1. a^2m^6.
2. c^4x^2.
3. $4p^{2n}r^{4n}$.
4. $\dfrac{e^4}{g^6}$.
5. $\dfrac{9h^8}{4p^2q^4}$.
6. $\dfrac{m^{2n}}{a^2c^{2m}}$.
7. $(a+b)^2(a-b)^2$.
8. $\dfrac{(a^2-x^2)^2}{(a^2+x^2)^2}$.

Express the cube roots of:

9. $\dfrac{8a^3m^9}{(a+b)^3}$.
10. $\dfrac{27(a+x)^3h^3}{8(a-x)^3m^9}$
11. $\dfrac{2^{3n} \cdot 3^{6n}}{(a+h)^{3n}}$.

Express the nth roots of:

12. x^n.
13. x^{2n}.
14. x^{mn}.
15. x^{n^2}.
16. x^{n^2+n}.
17. x^{n^2+2n}.
18. $\dfrac{x^{2n}}{c^{3n}}$.
19. $\dfrac{x^{mn}}{a^n}$.
20. $\dfrac{a^nx^{2n}}{c^n}$.

Fractional Exponents.

201. If we apply the rule of § 197 for division of exponents so as to express the square root of x^3, we shall have

$$\sqrt{x^3} = x^{\frac{3}{2}}.$$

Because $x = x^1$, we also have

$$\sqrt{x} = x^{\frac{1}{2}}.$$

Reversing the process and applying the general form,

$$x^m \times x^m = x^{2m},$$

we have
$$x^{\frac{1}{2}} \times x^{\frac{1}{2}} = x^{\frac{1}{2}+\frac{1}{2}} = x^1 = x,$$
$$x^{\frac{3}{2}} \times x^{\frac{3}{2}} = x^{\frac{3}{2}+\frac{3}{2}} = x^3.$$

Hence

A fractional exponent indicates the extraction of a root. If the denominator is 2, a square root is indicated; if 3, a cube root; if n, an nth root.

A fractional exponent has therefore the same meaning as the radical sign $\sqrt{\ }$, and may be used in place of it.

It is a mere question of convenience which method of indicating the root we shall use.

EXERCISES.

Express the following roots by exponents only:

1. $\sqrt{a}$.
2. $\sqrt{(a-b)}$.
3. $\sqrt{(a-b)^3}$.
4. $\sqrt{(a-b)^n}$.
5. $\sqrt{x^n}$.
6. $\sqrt{a^{3n}}$.
7. $\sqrt{c^2m^3n^5}$.
8. $\sqrt{pq^2r^3}$.
9. $\sqrt{\dfrac{ax^3}{cy^2}}$.
10. $\sqrt[3]{c^3m^2}$.
11. $\sqrt[3]{c^3m}$.
12. $\sqrt[3]{m^2x^3}$.
13. $\sqrt[3]{\dfrac{x^3}{b^2}}$.
14. $\sqrt[3]{\dfrac{(a-b)^2}{(a+b)^3}}$.
15. $\sqrt[3]{\dfrac{cx^3}{py^3}}$.
16. $\sqrt[3]{\dfrac{(a+b)x^3}{(a-b)y^3}}$.
17. $\sqrt[3]{\dfrac{(a+x)^2x^{3n}}{(a-x)^2y^{3n}}}$.
18. $\sqrt{\dfrac{(a+x)(a-x)}{(b+x)(b-x)}}$.

Express the nth roots of:

19. x.
20. x^2.
21. x^3.
22. $5mx^n$.
23. $7c^2x^{2n}$.
24. $10c^3x^{3n}$.

Powers of Expressions with Fractional Exponents.

202. THEOREM. *The pth power of the nth root is equal to the nth root of the pth power.*

EXAMPLE.
$$(\sqrt[3]{8})^2 = 2^2 = 4,$$
$$\sqrt[3]{8^2} = \sqrt[3]{64} = 4;$$

or, in other words, the square of the cube root of 8 (that is, the square of 2) is the cube root of the square of 8 (that is, of 64).

General Proof. Let us put $x =$ the nth root of a.

The pth power of this root x will then be x^p. (1)

Because x is the nth root of a,

$$x^n = a.$$

Raising both sides of this equation to the pth power, we have

$$x^{np} = a^p = p\text{th power of } a.$$

The nth root of the first member is found by dividing the exponent by n, which gives

$$n\text{th root of } p\text{th power} = x^p,$$

the same expression (1) just found for the pth power of the nth root.

203. This theorem leads to the following corollaries:

Cor. 1. *The expression*

$$a^{\frac{p}{n}}$$

may mean either the pth power of $a^{\frac{1}{n}}$ *or the nth root of* a^p, *these quantities being equivalent.*

Cor. 2. *The numerator of a fractional exponent is the index of a power. The denominator is the index of a root.*

Cor. 3. *The powers of expressions having fractional exponents may be formed by multiplying the exponents by the index of the power.*

EXERCISES.

Express the squares, the cubes and the nth powers of the following expressions:

1. $c^{\frac{1}{2}}$. 2. $c^{\frac{1}{3}}$. 3. $c^{\frac{3}{2}}$.

4. $ac^{\frac{1}{2}}$. 5. $a^{\frac{1}{2}}c^{\frac{1}{3}}$. 6. $a^{\frac{1}{2}}c^{\frac{1}{n}}$.

7. $a^{\frac{2}{n}}y^{\frac{3}{n}}$. 8. $c^{\frac{3}{n}}y^{\frac{4}{n}}$. 9. $c^{\frac{1}{n}}y^{\frac{2}{n}}$.

10. $\dfrac{m^{\frac{1}{2}}x}{a^{\frac{1}{3}}}$ 11. $\dfrac{n^{\frac{1}{3}}y}{b^{\frac{1}{n}}}$. 12. $\dfrac{n^{\frac{1}{n}}y^{\frac{1}{2}}}{a+b}$.

13. $\dfrac{(a+b)^{\frac{1}{2}}}{(a-b)^{\frac{n}{2}}}$. 14. $\dfrac{(m+n)^{\frac{1}{n}}}{(m-n)^{\frac{2}{n}}}$. 15. $\dfrac{(m+n)^{\frac{m}{n}}}{(m-n)^{\frac{1}{n}}}$.

204. ***Multiplication of Fractional Exponents.*** **One fractional exponent may be multiplied by another when a root is to be extracted, according to the rules for the multiplication of fractions.**

The process is the same as that of § 193 except that the exponent is fractional instead of entire, and roots are indicated as well as powers.

Example 1. Square root of $a^{\frac{1}{3}} = (a^{\frac{1}{3}})^{\frac{1}{2}} = a^{\frac{1}{6}}$.

Ex. 2. Cube root of $m^{\frac{2}{3}} = (m^{\frac{2}{3}})^{\frac{1}{3}} = m^{\frac{2}{9}}$.

Ex. 3. Square root of $(a+x)^{\frac{1}{2}}b^{\frac{3}{4}} = [(a+x)^{\frac{1}{2}}b^{\frac{3}{4}}]^{\frac{1}{2}} = (a+x)^{\frac{1}{4}}b^{\frac{3}{8}}$.

EXERCISES.

Express the square roots of:

1. $x^{\frac{1}{2}}$.
2. $a^{\frac{1}{3}}x^{\frac{1}{4}}$.
3. $(a+b)^{\frac{1}{3}}x^{\frac{2}{3}}$.
4. $\dfrac{x^{\frac{1}{3}}}{a^{\frac{1}{2}}}$.
5. $\dfrac{(a+b)^{\frac{2}{3}}}{(a-b)^{\frac{1}{3}}}$.
6. $\dfrac{4m^2x^{\frac{3}{4}}}{n}$.

Express the following indicated roots:

7. $(a^{\frac{1}{2}}b^{\frac{1}{2}})^{\frac{1}{3}}$. Ans. $a^{\frac{1}{6}}b^{\frac{1}{6}}$.
8. $(a^{\frac{1}{2}}b^{\frac{1}{2}})^{\frac{2}{3}}$. Ans. $a^{\frac{1}{3}}b^{\frac{1}{3}}$.
9. $(m^{\frac{1}{3}}n^{\frac{2}{3}})^{\frac{1}{2}}$.
10. $(m^{\frac{1}{2}}n^{\frac{1}{3}})^{\frac{2}{3}}$.
11. $\left((a+x)^{\frac{m}{n}}y^{\frac{1}{n}}\right)^{\frac{n}{m}}$.
12. $(a^{\frac{1}{2}}b^{\frac{1}{3}}c^{\frac{1}{4}})^{\frac{3}{2}}$.
13. $\left(g^{\frac{1}{n}}h^{\frac{2}{n}}\right)^{\frac{n}{2}}$.
14. $(c^2d^{\frac{3}{2}}x^{\frac{5}{2}})^{\frac{1}{3}}$.
15. $(pq^{\frac{1}{2}}x^2)^{\frac{1}{3}}$.
16. $\left(m^{\frac{m}{n}}n^{\frac{2}{n}}\right)^{\frac{n}{2}}$.
17. $\left(\dfrac{a^{\frac{2}{3}}}{b^{\frac{3}{2}}}\right)^{\frac{3}{2}}$.
18. $\left(\dfrac{x^m}{a^m}\right)^{\frac{1}{m}}$.
19. $\left(\dfrac{am^{\frac{n}{m}}}{ec^n}\right)^{\frac{m}{n}}$.

Negative Exponents.

205. The meaning of a negative exponent may be defined by the formula of § 45:

$$\frac{a^m}{a^n} = a^{m-n}. \qquad (1)$$

Let us suppose $m = 3$ and $n = 5$. The equation then becomes

$$\frac{a^3}{a^5} = a^{3-5} = a^{-2},$$

or, reducing,

$$\frac{1}{a^2} = a^{-2}. \qquad (2)$$

We hence conclude:

A negative exponent indicates the reciprocal of the corresponding quantity with a positive exponent.

Multiplying (2) by a^2 gives

$$a^2 \times a^{-2} = 1.$$

Dividing by a^{-2}, we conclude

$$\frac{1}{a^{-2}} = a^2.$$

Hence

A factor may be transferred from one term of a fraction to the other if the sign of its exponent be changed.

206. If, in the formula (1), we suppose $m = n$, it becomes

$$a^0 = 1.$$

Hence, because a may be any quantity whatever,

Any quantity with the exponent 0 *is equal to unity.*

This result may be made more clear by successive divisions of a power of a by a. Every time we effect this division, we subtract 1 from the exponent, and we may suppose this subtraction to continue to negative values of the exponent. In the left-hand members of the equations in the margin, the division is effected symbolically by diminishing the exponents; in the right-hand members the result is written out.

$$\begin{array}{lcl} a^3 & = & aaa \\ a^2 & = & aa \\ a^1 & = & a \\ a^0 & = & 1 \\ a^{-1} & = & \dfrac{1}{a} \\ a^{-2} & = & \dfrac{1}{aa} \\ \text{etc.} & & \text{etc.} \end{array}$$

EXERCISES.

Transform the following expressions by introducing negative exponents:

1. $\dfrac{a^2}{b^2}$. Ans. a^2b^{-2}.
2. $\dfrac{a^2}{b^{\frac{1}{2}}}$.
3. $\dfrac{a^2b^3}{m^{\frac{1}{2}}x^2}$.
4. $\left(\dfrac{ab}{mn^2}\right)^2$ Ans. $a^2b^2m^{-2}n^{-4}$.
5. $\left(\dfrac{mx^2}{pq^{\frac{1}{2}}}\right)^2$.
6. $\left(\dfrac{xy^2}{mn}\right)^3$.
7. $\dfrac{a}{b}$.
8. $\dfrac{x+h}{x-h}$. Ans. $(x+h)(x-h)^{-1}$.
9. $\left(\dfrac{x+h}{x-h}\right)^{\frac{1}{2}}$.
10. $\left(\dfrac{a+h}{(a-h)^2}\right)^{\frac{1}{2}}$.
11. $\left(\dfrac{m^2(m+n)^{\frac{1}{2}}}{n(m-n)}\right)^n$.
12. $\left(\dfrac{ab^3}{xy^2z}\right)^{\frac{3}{2}}$.

Express the following with positive exponents, reducing to fractions where necessary:

13. a^{-1}.
14. $\dfrac{1}{a^{-1}}$.
15. a^{-n}.

16. $\frac{1}{a^{-n}}$.
17. ab^{-1}.
18. $\frac{c}{a^{-n}}$.
19. $\frac{m^2}{m^{-3}}$.
20. $a^{-1}b$.
21. $\frac{m^{-2}}{m^{-1}}$.
22. a^nb^{-n}.
23. $a^{-1}xb^{-2}$.
24. $m^{-2}x^2n^{-1}$.
25. $(a+b)^{-1}$.
26. $(a+x)(a+x)^{-1}$.
27. $\frac{a+x}{(a+x)^{-1}}$.
28. $\frac{(a+x)^{-2}}{(a-x)^{-3}}$.
29. $\frac{a^{-1}}{b^{-1}}$.
30. $\frac{1}{1+x^{-1}}$.

207. The rules of involution and evolution by exponents apply to negative exponents, regard being paid to algebraic signs.

EXAMPLES.
$$\frac{1}{a^{-2}} = \frac{1}{\frac{1}{a^2}} = a^2.$$

$$(a^{-2})^{-3} = \frac{1}{(a^{-2})^3} = \left(\frac{1}{a^{-2}}\right)^3 = (a^2)^3 = a^6,$$

or

$$(a^{-2})^{-3} = \left(\frac{1}{a^2}\right)^{-3} = \frac{1}{(a^2)^{-3}} = (a^2)^3 = a^6.$$

Because $-2 \times -3 = +6$, the sign of the final exponent corresponds to the rule of signs in multiplication.

EXERCISES.

Free the following exponential expressions from parentheses and negative exponents:

1. $(m^{-2})^3$. Ans. $\frac{1}{m^6}$.
2. $(m^{-3})^2$.
3. $(m^{-3})^{-3}$.
4. $(ak^2)^{-3}$
5. $(g^{-1}k)^2$.
6. $(g^{-1}k)^{-2}$.
7. $(n^{-2}k^3)^{-1}$.
8. $(h^{-1}x^{-1})^{-1}$.
9. $(h^{-\frac{1}{2}}x^{-\frac{1}{2}})^2$.
10. $(p^{\frac{1}{2}}g^{-\frac{1}{2}})^2$.
11. $(p^{-\frac{1}{2}}g^{\frac{1}{2}})^{-2}$.
12. $(p^2g^{-2})^{-\frac{1}{2}}$.
13. $(p^{\frac{1}{2}}y^{\frac{1}{2}})^{-\frac{1}{2}}$.
14. $(p^{-1}x^{\frac{1}{2}})^2$.
15. $(m^{\frac{1}{3}}y^{-\frac{1}{3}})^{-6}$.
16. $\left(\frac{a^{-2}}{b^{-2}}\right)^{-1}$.
17. $\left(\frac{a^2c^{-2}}{bc}\right)^{-2}$
18. $\left(\frac{mn}{m^{-1}n^{-1}}\right)^{-2}$.

SECTION II. INVOLUTION AND EVOLUTION OF POLYNOMIALS.

The Binomial Theorem.

208. Let us form the successive powers of the binomial $1 + x$. We multiply according to the method of § 121:

$$
\begin{array}{lrl}
 & (1+x)^1 = & 1 + x \\
\text{Multiplier,} & & \underline{1 + x} \\
 & & 1 + x \\
 & & \underline{\quad + x \ + x^2} \\
 & (1+x)^2 = & 1 + 2x + x^2 \\
\text{Multiplier,} & & \underline{\quad 1 \quad + x} \\
 & & 1 + 2x + x^2 \\
 & & \underline{\qquad x + 2x^2 + x^3} \\
 & (1+x)^3 = & 1 + 3x + 3x^2 + x^3 \\
\text{Multiplier,} & & \underline{\qquad\qquad\qquad 1 + x} \\
 & & 1 + 3x + 3x^2 + x^3 \\
 & & \underline{\qquad x + 3x^2 + 3x^3 + x^4} \\
 & (1+x)^4 = & 1 + 4x + 6x^2 + 4x^3 + x^4.
\end{array}
$$

It will be seen that whenever we multiply one of these powers by $1 + x$, the coefficients of x, x^2, etc., which we add to form the next higher power, are the same as those of the given power, only those in the lower line go one place toward the right. Thus, to form $(1 + x)^4$, we took the coefficients of $(1 + x)^3$, and wrote and added them thus:

Coef. of $(1 + x)^3$,	1,	3,	3,	1		
		1,	3,	3,	1	
Coef. of $(1 + x)^4$,	1,	4,	6,	4,	1	
		1,	4,	6,	4,	1
Again adding,		1,	5,	10,	10,	5, 1

And so on to any extent.

It is not necessary to write the numbers under each other to add them in this way; we have only to add each number to the one on the left in the same line to form the corresponding number of the line below. Thus we can form the coefficients of the successive powers of x at sight as in the following table. The first figure in each line is 1; the next is the coefficient of x; the third the coefficient of x^2, etc.

For practice let the pupil continue this table to $n = 10$.

$n = 1$;	coefficients,	1,	1.						
$n = 2$;	"	1,	2,	1.					
$n = 3$;	"	1,	3,	3,	1.				
$n = 4$;	"	1,	4,	6,	4,	1.			
$n = 5$;	"	1,	5,	10,	10,	5,	1.		
$n = 6$;	"	1,	6,	15,	20,	15,	6,	1.	
etc.					etc.				

It is evident that the first quantity is always 1, and that the next coefficient in each line, or the coefficient of x, is n.

The third is not evident, but is really equal to

$$\frac{n(n-1)}{2}, \qquad (a)$$

as will be readily found by trial; because, beginning with $n = 3$,

$$3 = \frac{3 \cdot 2}{2}, \quad 6 = \frac{4 \cdot 3}{2}, \quad 10 = \frac{5 \cdot 4}{2}, \quad \text{etc.}$$

The fourth number on each line is

$$\frac{n(n-1)(n-2)}{2 \cdot 3}. \qquad (b)$$

Thus, beginning as before with the third line, where $n = 3$,

$$1 = \frac{3 \cdot 2 \cdot 1}{2 \cdot 3}, \quad 4 = \frac{4 \cdot 3 \cdot 2}{2 \cdot 3}, \quad 10 = \frac{5 \cdot 4 \cdot 3}{2 \cdot 3}, \quad \text{etc.} \qquad (c)$$

209. The numbers given by the formulæ (a), (b), (c), etc., are called **binomial coefficients.** In writing them, we may multiply all the denominators by the factor 1 without changing them, so that there will be as many factors in the denominator as in the numerator. The fourth column of coefficients (c) will then be written

$$\frac{3 \cdot 2 \cdot 1}{1 \cdot 2 \cdot 3}, \quad \frac{4 \cdot 3 \cdot 2}{1 \cdot 2 \cdot 3}, \quad \frac{5 \cdot 4 \cdot 3}{1 \cdot 2 \cdot 3}, \quad \text{etc.}$$

We can find all the binomial coefficients of any power when we know the value of n.

In every binomial coefficient the first factor in the numerator is n, the second $n - 1$, etc., each factor being less by unity than the preceding one.

The first factor in the denominator is 1, the second 2, the third 3, etc.

The numerator and denominator of the second coefficient will contain two factors, as in (a); of the third, three factors, as in (b) and (c); of the fourth, four factors, etc.

EXERCISES.

1. Form all the binomial coefficients for the fifth power of $1 + x$.

$$\text{Ans. } \frac{5}{1} = 5; \ \frac{5.4}{1.2} = 10; \ \frac{5.4.3}{1.2.3} = \frac{5.4}{1.2} = 10;$$
$$\frac{5.4.3.2}{1.2.3.4} = \frac{5}{1} = 5; \ \frac{5.4.3.2.1}{1.2.3.4.5} = 1.$$

It will be seen that after we reach the middle of the series of coefficients the last factors begin to cancel the preceding ones, so that the numbers repeat themselves in reverse order.

2. Form all the binomial coefficients for

$$n = 6; \quad n = 7; \quad n = 8; \quad n = 9.$$

210. The conclusion reached in the preceding section is embodied in the following equation, which should be perfectly memorized:

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{1.2}x^2 + \frac{n(n-1)(n-2)}{1.2.3}x^3 + \frac{n(n-1)(n-2)(n-3)}{1.2.3.4}x^4 + \ldots + x^n.$$

211. *Def.* When a single expression is changed into the sum of a series of terms, it is said to be **developed**, and the series is called its development.

EXAMPLE. The preceding series $1 + nx + \frac{n(n-1)}{1.2}x^2 +$ etc., is the development of $(1 + x)^n$.

EXERCISES.

Develop the following expressions:

1. $(1 + a)^4$. 2. $(1 + ax)^3$. 3. $\left(1 + \frac{m}{p}\right)^5$.

4. $(1 + h^2)^2$. 5. $(1 + h^2)^5$. 6. $(1+ah^3)^4$.

212. If the second term of the binomial is negative, its odd powers will be negative (§194). Hence the signs of the alternate terms of the development will then be changed as in the following form:

$$(1-x)^n = 1 - nx + \frac{n(n-1)}{1.2}x^2 - \frac{n(n-1)(n-2)}{1.2.3}x^3 + \text{etc.}$$

EXERCISES.

Develop:

1. $(1-b)^3$.
2. $(1-b)^4$.
3. $(1-b)^5$.
4. $(1-ah)^4$.
5. $(1-ah^2)^5$.
6. $\left(1-\frac{m^2}{n}\right)^6$.

213. The development of any binomial may be reduced to the preceding form by a transformation. Let us put

$a \equiv$ the first term of the binomial;
$b \equiv$ the second term;
$n \equiv$ the index of the power.

The binomial will then be $a+b$, which we may write in either of the forms

$$a\left(1+\frac{b}{a}\right) \quad \text{or} \quad b\left(1+\frac{a}{b}\right).$$

The nth power of the first form will be

$$a^n\left(1+\frac{b}{a}\right)^n.$$

By § 210,

$$\left(1+\frac{b}{a}\right)^n = 1 + n\frac{b}{a} + \frac{n(n-1)}{1\,.\,2}\frac{b^2}{a^2} + \frac{n(n-1)(n-2)}{1\,.\,2\,.\,3}\frac{b^3}{a^3} + \text{etc.}$$

Multiplying by a^n, we have

$$a^n\left(1+\frac{b}{a}\right)^n = (a+b)^n$$

$$= a^n + na^{n-1}b + \frac{n(n-1)}{1\,.\,2}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{1\,.\,2\,.\,3}a^{n-3}b^3 + \text{etc.}$$

We see from this that, in the development of $(a+b)^n$,

The first term is a^nb^0 (because $b^0 = 1$).

The second term is $a^{n-1}b^1$, and its coefficient is the index of the power.

In each successive term the exponent of a is diminished

and that of b increased by unity, while the coefficients are the same as in the case of $(1+x)^n$.

REMARK. The work of developing a power of a binomial is facilitated by the following arrangement:

1. Write in one line all the powers of the first term, beginning with the highest and ending with the zero power, or unity.

2. Write under them the corresponding powers of the second term, beginning with the zero power, or unity, and ending with the highest.

3. Under this, in a third line, write the binomial coefficients.

4. Form the continued product of each column of three factors, and by connecting with the proper signs we shall have the required power.

EXAMPLE. Form the sixth power of $2a - 3x^2$.

Powers of $2a$,	$64a^6$	$+\ 32a^6$	$+\ 16a^4$	$+\ 8a^3$	$+\ 4a^2$	$+\ 2a$	$+\ 1$
" " $-3x^2$,	1	$-\ 3x^2$	$+\ 9x^4$	$-\ 27x^6$	$+\ 81x^8$	$-\ 243x^{10}$	$+\ 729x^{12}$
Binom. coef.,	1	$+\ 6$	$+\ 15$	$+\ 20$	$+\ 15$	$+\ 6$	$+\ 1$

$$(2a-3x^2)^6 = 64a^6 - 576a^6x^2 + 2160a^4x^4 - 4320a^3x^6 + 4860a^2x^8 - 2916ax^{16} + 729x^{12}.$$

EXERCISES.

Develop:

1. $(a+b)^3$.
2. $(a+b)^4$.
3. $(a+b)^7$.
4. $\left(a+\frac{1}{a}\right)^3$.
5. $(a-a^{-1})^4$.
6. $\left(a^2+\frac{n}{a^2}\right)^3$.
7. $(a+2x^2)^6$.
8. $(c-3h^2)^5$.
9. $\left(m-\frac{4}{n}\right)^4$.
10. $(c^2-x^2)^5$.
11. $\left(a+\frac{2}{a}\right)^4$.
12. $\left(1+\frac{n}{2}\right)^5$.
13. $(a^{-1}+b^{-2})^6$.
14. Write the first five terms of $(m+n)^m$.
15. " " " " " " $(a+b)^{2n}$.
16. " " " " " " $(a-b)^{\frac{n}{2}}$.
17. " " " " " " $(a-2x)^{\frac{n}{2}}$.
18. " " " " " " $(a+3x)^{\frac{n}{3}}$.
19. " " " " " " $\left(1+\frac{1}{n}\right)^n$.

Square Root of a Polynomial.

214. Sometimes the square root of a polynomial can be extracted. To extract a root, the polynomial must be arranged according to the powers of some one symbol. Suppose x to be that symbol. Then supposing the *root* arranged according to the powers of x, calling n the index of the highest power of x, and calling a, b, c, etc., the coefficients of the powers of x, we shall have

$$\text{Root} = ax^n + bx^{n-1} + \ldots + l.$$

Squaring, we find that

$$a^2x^{2n}$$

will be the highest term of the square, and

$$l^2 \quad .$$

the lowest term. Hence

A polynomial can have no root unless both its highest and lowest terms are perfect squares.

215. If the preceding condition is fulfilled, *perhaps* the polynomial has a root. If it has, the root is found by the following

RULE. 1. *Arrange the polynomial according to powers of that symbol of which the powers are most numerous.*

2. *Extract the square root of the highest term, and subtract its square from the polynomial.*

3. *Find the next term of the root by dividing the second term of the polynomial by double the term of the root already found.*

4. *Multiply twice the first term plus the second term by the second term, and subtract the product.*

5. *Find each term in succession by dividing the first term of the remainder by the first divisor.*

6. *Multiply every new term by twice the part of the root already found plus this term, and subtract the product.*

If there is any root, the last remainder will come out zero.

216. *Reason of the Rule.* To show the truth of this rule let us take the polynomial

$$x^4 + 2(m+n)x^3 + (m^2+n^2)x^2 - 2mn(m+n)x + m^2n^2. \quad (a)$$

The square root of the first term is x^2. Let us put P for the sum of the remaining terms, so that the root is

$$x^2 + P. \quad (b)$$

Then, by squaring,

$(x^2+P)^2 = x^4+2Px^2+P^2 = x^4+2(m+n)x^3+(m^2+n^2)x^2 - \text{etc.},$

or, dropping x^4 fron each member,

$$2Px^2 + P^2 = 2(m+n)x^3 + (m^2+n^2)x^2 + \text{etc.} \qquad (c)$$

Since these two members are to be identically equal, the highest terms must be equal; that is, the highest term of $2Px^2$ must be

$$2(m+n)x^3,$$

or, dividing by $2x^2$,

$$\text{Highest term of } P = (m+n)x.$$

That is, *we find the second term of the root by dividing the second term of the polynomial by twice the first term of the root.* This is No. 3 of the rule.

Next, by the rule, we multiply twice the first term *plus* the second term by the second term, and subtract the product. That is, if we put, for shortness,

$$a \equiv x^2, \text{ the first term;}$$

$$b \equiv (m+n)x, \text{ the second term;}$$

the quantity the rule tells us to subtract is

$$(2a+b)b = 2ab + b^2.$$

But, by No. 2 of the rule, we have already subtracted $a^2 \equiv x^4$, the square of the first term of the root. Hence we have subtracted in all

$$a^2 + 2ab + b^2 = (a+b)^2,$$

which is the square of the sum of the first two terms of the root.

Reasoning in the same way, we find that, at each step, *the total quantity subtracted from the polynomial is the square of the sum of all the terms of the root already found.*

Hence if at any step we have zero for a remainder, it will show that the square of the sum of the terms of the root is equal to the polynomial, and therefore that the root is really the root of the polynomial.

The arrangement of the work is shown in the following examples, which the student should perform, and by which he should show that the sum of the quantities subtracted at each step is really the square of that part of the root already found.

Example 1.

$$
\begin{array}{r|l|l}
 & & x^2+(m+n)x-mn \\
x^2 & x^4+2(m+n)x^3+(m^2+n^2)x^2-2mn(m+n)x+m^2n^2 & \\
x^2 & x^4 & \\
\hline
2x^2+(m+n)x & 2(m+n)x^3+(m^2+n^2)x^2 & \\
\text{Mult. } (m+n)x & 2(m+n)x^3+(m^2+2mn+n^2)x^2 & \\
\hline
2x^2+2(m+n)x-mn & -2mnx^2-2mn(m+n)x+m^2n^2 & \\
\text{Multiplier, } -mn & -2mnx^2-2mn(m+n)x+m^2n^2 & \\
\hline
\end{array}
$$

The order of operations is this: We find the term x^2 of the root by extracting the root of the highest term. We write this term three times: first on the right, as a part of the root, and then on the left, as a trial divisor, and again as a multiplier of this divisor.

Multiplying $x^2 \times x^2$ we have x^4 to be subtracted from the given expression. We then bring down two more terms.

Forming $x^2 + x^2 = 2x^2$, the latter is a trial divisor, which, divided into the first term brought down, gives $(m + n)x$ as the second term of the root. We write this term three times: first on the right, then after $2x^2$, and then under itself as a multiplier.

Multiplying the two terms already found by $(m + n)x$, we subtract the product from the terms brought down, and thus find a second remainder, to which we bring down the remaining terms of the expression.

Adding $(m+n)x$ to $2x^2+(m+n)x$, we have $2x^2+ 2(m+n)x$ as the third trial divisor. Dividing the first term by $2x^2$, we have the term $-mn$ of the root, with which we proceed as before.

Example 2.

$$
\begin{array}{r|l|l}
x^2 & x^4 - 2ax^3 + (a^2 + 4b^2)x^2 - 4ab^2x + 4b^4 & x^2 - ax + 2b^2 \\
x^2 & x^4 & \\
\hline
2x^2 - ax & -2ax^3 + (a^2 + 4b^2)x^2 & \\
-ax & -2ax^3 + a^2x^2 & \\
\hline
2x^2 - 2ax + 2b^2 & 4b^2x^2 - 4ab^2x + 4b^4 & \\
2b^2 & 4b^2x^2 - 4ab^2x + 4b^4 & \\
\hline
 & 0 \qquad 0 \qquad 0 & \\
\end{array}
$$

EXERCISES.

Extract the square roots of:

1. $a^4 - 2a^3 + 3a^2 - 2a + 1$.
2. $a^4 - 4a^3 + 8a + 4$.
3. $4x^4 - 12ax^3 + 25a^2x^2 - 24a^3x + 16a^4$.
4. $x^6 - 6ax^5 + 15a^2x^4 - 20a^3x^3 + 15a^4x^2 - 6a^5x + a^6$.
5. $25x^4 - 30ax^3 + 49a^2x^2 - 24a^3x + 16a^4$.
6. $a^4 + (4b - 2c)a^3 + (4b^2 - 4bc + 3c^2)a^2 + (4bc^2 - 2c^3)a + c^4$.
7. $a^2b^2 + a^2c^2 + b^2c^2 + 2a^2bc - 2ab^2c - 2abc^2$.
8. $9x^4 + 12x^3 + 10x^2 + 4x + 1$.
9. $9x^{2n} + 12x^{\frac{3n}{2}} + 10x^n + 4x^{\frac{n}{2}} + 1$.

Remark. The root may be extracted by beginning with the lowest term as well as with the highest. The pupil should perform the above exercises in both ways. The following is the solution of Ex. 6, when we begin with the lowest term in a:

$$
\begin{array}{r|l}
 & c^2 + (2b - c)a + a^2 \\
\hline
c^2 & c^4 + (4bc^2 - 2c^3)a + (4b^2 - 4bc + 3c^2)a^2 + (4b - 2c)a^3 + a^4 \\
c^2 & c^4 \\
\hline
2c^2 + (2b - c)a & (4bc^2 - 2c^3)a + (4b^2 - 4bc + 3c^2)a^2 \\
(2b - c)a & (4bc^2 - 2c^3)a + (4b^2 - 4bc + c^2)a^2 \\
\hline
2c^2 + (4b - 2c)a + a^2 & 2c^2a^2 + (4b - 2c)a^3 + a^4 \\
+ a^2 & 2c^2a^2 + (4b - 2c)a^3 + a^4 \\
\hline
 & 0 \qquad 0 \qquad 0
\end{array}
$$

217. It is only in special cases that the root of a polynomial can be extracted. But when it cannot be extracted, we may continue the process to any extent, though we shall never get zero for a remainder.

If the polynomial is arranged according to ascending powers of the leading symbol, the degree of the root in respect to that symbol will go on increasing by unity with every term added.

In the contrary case the degree will go on diminishing, finally becoming negative.

EXAMPLE. Extract the square root of $1 + x$.

$$
\begin{array}{l|l}
1 \quad 1 + x & 1 + \tfrac{1}{2}x - \tfrac{1}{8}x^2 + \tfrac{1}{16}x^3 - \tfrac{5}{128}x^4 + \tfrac{7}{256}x^5 - \text{etc.} \\
1 \quad 1 & \\
\hline
2 + \tfrac{1}{2}x & x \\
\quad + \tfrac{1}{2}x & x + \tfrac{1}{4}x^2 \\
\hline
2 + x - \tfrac{1}{8}x^2 & -\tfrac{1}{4}x^2 \\
\qquad - \tfrac{1}{8}x^2 & -\tfrac{1}{4}x^2 - \tfrac{1}{8}x^3 + \tfrac{1}{64}x^4 \\
\hline
2 + x - \tfrac{1}{4}x^2 + \tfrac{1}{16}x^3 & +\tfrac{1}{8}x^3 - \tfrac{1}{64}x^4 \\
\qquad\qquad + \tfrac{1}{16}x^3 & +\tfrac{1}{8}x^3 + \tfrac{1}{16}x^4 - \tfrac{1}{64}x^5 \text{ etc.} \\
\hline
2 + x - \tfrac{1}{4}x^2 & -\tfrac{5}{64}x^4 + \tfrac{1}{64}x^5 \text{ etc.} \\
 & -\tfrac{5}{64}x^4 - \tfrac{5}{128}x^5 \text{ etc.} \\
\hline
 & \tfrac{7}{128}x^5.
\end{array}
$$

EXERCISE.

Extract the square root of $1 - x$ by the above process, carrying the result to x^5.

Square Roots of Numbers.

218. The square root of a number may be extracted by a process similar to that of extracting the square root of a polynomial. But there are some points peculiar to a number to be understood.

219. *Number of Figures in the Root.* By studying the equations,

$$
\begin{aligned}
3^2 &= 9, \text{ or } \sqrt{9} = 3; \\
9^2 &= 81, \text{ or } \sqrt{81} = 9; \\
10^2 &= 100, \text{ or } \sqrt{100} = 10; \\
50^2 &= 2500, \text{ or } \sqrt{2500} = 50;
\end{aligned}
$$

we see that

If a number has 1 or 2 digits, its square root has 1 digit;
If a number has 3 or 4 digits, its square root has 2 digits;
If a number has 5 or 6 digits, its square root has 3 digits;
etc. etc.;

or, in general:

1. *The square root has as many digits as the number itself has pairs of digits, a single digit left over being counted as a pair.*

2. *The first figure of the root will be the square root of the greatest perfect square contained in the first figure or pair of figures of the number.*

220. *Case of Decimals.* If we study the equations

$$0.9^2 = 0.81, \text{ or } \sqrt{0.81} = 0.9;$$
$$0.3^2 = 0.09, \text{ or } \sqrt{0.09} = 0.3;$$
$$0.09^2 = 0.0081, \text{ or } \sqrt{0.0081} = 0.09;$$
$$0.03^2 = 0.0009, \text{ or } \sqrt{0.0009} = 0.03;$$

we see that:

1. *The first figure of the square root of a decimal is the square root of the greatest square contained in the first pair of figures of the decimal.*

2. *If the given decimal has two or more zeros after the decimal point, the decimal of the root will begin with as many zeros as the number begins with pairs of zeros.*

EXAMPLES AND EXERCISES.

How do the square roots of the following decimal fractions commence?

1. $\sqrt{0.223}$. Ans. 0.4. because 0.4 is the largest single digit whose square is contained in 0.22.

2. $\sqrt{0.033}$. Ans. 0.1, because 0.1 is the largest single digit whose square is not greater than 0.03.

3. $\sqrt{0.729}$. 4. $\sqrt{0.053}$. 5. $\sqrt{0.0092}$.

6. $\sqrt{0.0005}$. 7. $\sqrt{0.00032}$. 8. $\sqrt{0.00008}$.

221. *Extraction of the Square Root.* Let us put

$N \equiv$ the number whose root is to be extracted.

$n \equiv$ the value of the first figure of the root.

This value may be n units, n tens, n hundreds, etc., according to the place it occupies.

$P \equiv$ the value of all the figures after the first.

$Q \equiv$ the value of all after the second, etc.

Then we have, by supposition,

$$\text{Root} = \sqrt{N} = n + P.$$

etc. etc.

Squaring, we have

$$N = (n + P)^2 = n^2 + 2nP + P^2;$$

whence

$$2nP + P^2 = N - n^2,$$

or

$$(2n + P)\ P = N - n^2.$$

Dividing by P, we have

$$P = \frac{N - n^2}{2n + P}.$$

From this equation we have to find, by trial, the first figure of P, when we shall have the first two figures of the root. Repeating the process, we shall find a third figure, etc. The general rule will be:

RULE. 1. *Separate the figures of the number into pairs, starting from the decimal point or unit's place.*

2. *The first figure of the root is the greatest number whose square is contained in the first pair or figure of the number.*

3. *Subtract the square of the first figure of the root from the first pair or figure of the number, and bring down the next pair.*

4. *Divide this result by twice the first figure of the root as a trial divisor; add the quotient to the trial divisor, multiply the sum by the quotient and subtract from the dividend. For the quotient must be taken the largest number which will give a product less than the dividend.*

5. *After subtracting this product, bring down another pair of figures, double the part of the root already found, and repeat the process until as many figures of the root as are wanted are found.*

The reason of this rule is nearly the same as that of the rule for polynomials. If we call n, p, q, etc., the values of the successive figures of the root, the sum of the quantities we subtracted is the square of $n + p + q +$ etc.

We first point the given number off into pairs of digits from the unit's place, as is commonly explained in arithmetic. We may then have, on the left, either one figure or two. The largest square which it contains is the first figure of the root. We subtract the square of this figure, and use double the root figure as a trial divisor. This gives 1 as the

second figure, which we affix to the trial divisor and also use as a multiplier. Subtracting the product again, we use double the part of the root as a second trial divisor, and so on.

After getting a certain number of figures, we may find nearly as many more without changing the divisor. We may then cut off one figure from the divisor for every one we add to the root.

As an example, let us find the square root of 26890348.

```
           26890348 | 5185.5904196
           25
       101) 189
         1  101
      1028)  8803
         8   8224
     10365)   57948
          5   51825
   10370.5)    6123.00
          5    5185.25
  10371.09)     937.7500
          9     933.3981
10|3|7|1.|1|8)    4.3519
                  4.1485
                    2034
                    1037
                     997
                     933
                      64
                      62
                       2
```

EXERCISES.

Extract the square roots of:

1. 2193.
2. 46084.
3. 293462.
4. 89189219.
5. 82.6292.
6. 41693.

SECTION III. OPERATIONS UPON IRRATIONAL EXPRESSIONS.

Definitions.

222. *Def.* A **rational expression** is one in which the only indicated operations are those of addition, subtraction, multiplication or division.

All the operations we have hitherto considered, except the extraction of roots, have led to rational expressions.

Def. A **perfect square, cube** or nth **power** is an expression of which the square root, cube root or nth root can be extracted.

223. *Def.* An expression which requires the extraction of a root is called **irrational.**

EXAMPLE. Irrational expressions are

$$\sqrt{a}, \qquad \sqrt[3]{(a+b)}, \qquad \sqrt{27};$$

or, in the language of exponents,

$$a^{\frac{1}{2}}, \qquad (a+b)^{\frac{1}{3}}, \qquad 27^{\frac{1}{2}}.$$

Def. Expressions irrational in form are called **irreducible** when incapable of being expressed without the radical sign.

REMARK. Only irreducible expressions are properly called irrational.

EXAMPLE. The expressions

$$(a^2 + 2ab + b^2)^{\frac{1}{2}}, \qquad \sqrt{36},$$

though irrational in form are not so in reality, because they are equal to $a + b$ and 6 respectively, which are rational.

Def. A **surd** is an indicated root which cannot be extracted exactly.

Def. A **quadratic surd** is a surd with an indicated square root.

EXAMPLE. The expression $a + b\sqrt{x}$ is irrational, and the surd is $\sqrt{x}$, which is a quadratic surd.

Def. Irrational terms are **similar** when they contain identical surds.

EXAMPLES. The terms $\sqrt{30}$, $7\sqrt{30}$, $(x+y)\sqrt{30}$ are similar, because the quantity under the radical sign is 30 in each.

The terms $(a+b)\sqrt{x+y}$, $3\sqrt{x+y}$, $m\sqrt{x+y}$ are similar.

Aggregation of Similar Surds.

224. Irrational terms may be aggregated by the rules of §§ 28 and 30, the surds being treated as if they were single symbols. Hence

When similar irrational terms are connected by the signs + or −, the coefficients of the similar surds may be aggregated, and the surd itself affixed to their sum.

EXAMPLE. The sum

$$a\sqrt{(x+y)} - b\sqrt{(x+y)} + 3\sqrt{(x+y)}$$

may be transformed into $(a-b+3)\sqrt{(x+y)}$.

EXERCISES.

Reduce the following expressions to the smallest number of terms:

1. $9\sqrt{a} - 5ab\sqrt{2} + 6\sqrt{a} + 7ab\sqrt{2}$.
2. $6a^{\frac{1}{2}} - 7b^{\frac{1}{3}} + 8a^{\frac{1}{2}} + 9b^{\frac{1}{3}}$.
3. $(a^2b)^{\frac{1}{2}} + (b^2a)^{\frac{1}{2}} - aa^{\frac{1}{2}} - bb^{\frac{1}{2}}$. Ans. $(a-b)b^{\frac{1}{2}} + (b-a)a^{\frac{1}{2}}$.
4. $(c^3x)^{\frac{1}{3}} + (b^3y)^{\frac{1}{3}} - bx^{\frac{1}{3}} - cy^{\frac{1}{3}}$.
5. $(a+b)\sqrt{x} + (a-b)\sqrt{x}$.
6. $a(x^{\frac{1}{2}} + y^{\frac{1}{2}}) + b(x^{\frac{1}{2}} - y^{\frac{1}{2}})$.
7. $a(x^{\frac{1}{2}} - y^{\frac{1}{2}}) + b(y^{\frac{1}{2}} - z^{\frac{1}{2}}) + c(z^{\frac{1}{2}} - x^{\frac{1}{2}})$.
8. $(d+c)\sqrt{x+y} - (d-c)\sqrt{x+y}$.
9. $7\sqrt{x+y} + 8\sqrt{x-y} - 5\sqrt{x+y} - 7\sqrt{x-y}$.
10. $\frac{1}{3}\sqrt{m+n} - \frac{1}{2}\sqrt{m-n} + \frac{1}{6}\sqrt{m+n} - \frac{1}{3}\sqrt{m-n}$.
11. $\frac{1}{2}(a+b)(\sqrt{x} + \sqrt{y}) - \frac{1}{2}(a-b)(\sqrt{x} - \sqrt{y})$.
12. $(m+n)(a^{\frac{1}{2}} - b^{\frac{1}{2}}) + (m-n)(a^{\frac{1}{2}} + b^{\frac{1}{2}})$.
13. $(m+n-2p)(a^{\frac{1}{2}} + b^{\frac{1}{2}} - c^{\frac{1}{2}}) + (m-2n+p)(a^{\frac{1}{2}} - b^{\frac{1}{2}} + c^{\frac{1}{2}})$
    $+ (-2m+n+p)(-a^{\frac{1}{2}} + b^{\frac{1}{2}} + c^{\frac{1}{2}})$.

Multiplication of Surds.

225. Irrational expressions may sometimes be transformed so as to have different expressions under the radical sign by the method of § 197, applying the following theorem:

THEOREM. *A root of the product of several factors is equal to the product of their roots.*

In the language of algebra,

$$\sqrt[n]{abcd}, \text{ etc.} = \sqrt[n]{a}\,\sqrt[n]{b}\,\sqrt[n]{c}\,\sqrt[n]{d}, \text{ etc.}$$
$$= a^{\frac{1}{n}}b^{\frac{1}{n}}c^{\frac{1}{n}}d^{\frac{1}{n}}, \text{ etc.}$$

Proof. By raising the members of this equation to the nth power, we shall get the same result, namely,

$$a \times b \times c \times d, \text{ etc.}$$

EXAMPLE. $\sqrt{36} = \sqrt{4}\,\sqrt{9}$,

which, by extracting the roots, becomes

$$6 = 2 \cdot 3,$$

a true equation.

EXERCISES

Prove the truth of the following equations by extracting the roots in both members:

$$\sqrt{4}\,\sqrt{49} = \sqrt{4.49}.$$
$$\sqrt{4}\,\sqrt{25} = \sqrt{4.25}.$$
$$\sqrt{4}\,\sqrt{36} = \sqrt{4.36}.$$
$$\sqrt{9}\,\sqrt{25} = \sqrt{9.25}.$$

226. *Application of the Theorem to the Multiplication of Surds.* The preceding theorem gives the following rule for finding the product of two or more surds with equal indices:

RULE. *Form the product of the quantities under the radical sign, and write the similar indicated root of the product.*

If this root can be exactly extracted, the product is rational.

EXERCISES.

Express each of the following products by a single surd, or without surds.

1. $\sqrt{x} \times \sqrt{y}$. Ans. $\sqrt{xy}$.
2. $\sqrt{5} \times \sqrt{a}$.
3. $\sqrt{a} \times \sqrt{b} \times \sqrt{c}$.
4. $\sqrt{mn} \times \sqrt{p} \times \sqrt{7}$.
5. $\sqrt{7} \times \sqrt{5}$.

6. $(a+b)^{\frac{1}{2}}(a-b)^{\frac{1}{2}}$. Ans. $(a^2-b^2)^{\frac{1}{2}}$.

7. $(x+y)^{\frac{1}{2}}(x-y)^{\frac{1}{2}}$.

8. $\sqrt{x+ay} \times \sqrt{x-ay}$.

9. $\sqrt{a^2+1} \times \sqrt{a+1} \times \sqrt{a-1}$.

10. $\sqrt{m} \times \sqrt{n} \times \sqrt{m+n}$.

11. $\sqrt{5} \times \sqrt{3} \times \sqrt{a+b}$.

12. $\sqrt{a} \times \sqrt{b} \times \sqrt{a+b} \times \sqrt{a-b}$.

13. $\sqrt{ab} \times \sqrt{bc} \times \sqrt{ca}$. Ans. abc.

14. $(2mn)^{\frac{1}{2}} \times (2mp)^{\frac{1}{2}} \times (np)^{\frac{1}{2}}$.

15. $2^{\frac{1}{2}}m^{\frac{1}{2}} \times 2^{\frac{1}{2}}n^{\frac{1}{2}} \times 2^{\frac{1}{2}}m^{\frac{1}{2}}$.

16. $2^{\frac{1}{2}}(a+b)^{\frac{1}{2}} \times 2^{\frac{1}{2}}(a-b)^{\frac{1}{2}} \times (a+b)^{\frac{1}{2}}$.

17. $(abc)^{\frac{1}{3}} \times (bcd)^{\frac{1}{3}} \times (cda)^{\frac{1}{3}} \times (dab)^{\frac{1}{3}}$.

18. $a\sqrt{x}\sqrt{y} \times b\sqrt{x}\sqrt{y} \times c\sqrt{x}\sqrt{y}$.

19. $a(a+b)^{\frac{1}{2}} \times (a-b)^{\frac{1}{2}} + b(a+b)^{\frac{1}{2}} \times (a-b)^{\frac{1}{2}}$.
Ans. $(a+b)(a^2-b^2)^{\frac{1}{2}}$.

20. $a\sqrt{x+y} \times \sqrt{x-y} - b\sqrt{x+y} \times \sqrt{x-y}$.

21. $\sqrt{2mn} \times \sqrt{2mx} \times \sqrt{nx} \times \sqrt{y}$.

22. $\sqrt{6} \times \sqrt{15} \times \sqrt{10}$.

23. $a^{\frac{1}{2}}\left(1+\frac{b}{a}\right)^{\frac{1}{2}}$.

24. $m^{\frac{1}{2}}\left(\frac{m}{n}-\frac{n}{m}\right)^{\frac{1}{2}}$.

25. $\sqrt{(m+n)^{\frac{1}{3}}} \times \sqrt{(m+n)^{\frac{5}{3}}}$

26. $\sqrt[3]{2} \times \sqrt[3]{2} \times \sqrt[3]{16}$.

27. $\sqrt[3]{(a+b)} \times \sqrt[3]{(a-b)}$.

28. $\sqrt[3]{\frac{m}{n}} \times \sqrt[3]{\frac{n}{m}}$.

29. $\sqrt[3]{\frac{1}{a+b}} \times \sqrt[3]{\frac{1}{(a+b)^2}}$.

30. $\sqrt[3]{\frac{m}{h}} \times \sqrt[3]{\frac{m^2}{h^2}}$.

31. $\sqrt[3]{\frac{a+b}{a-b}} \times \sqrt[3]{\frac{a}{a+b}}$.

32. $\sqrt{\frac{1}{h+k}} \times \sqrt{\frac{1}{h-k}} \times \sqrt{\frac{q}{h+k}} \times \sqrt{\frac{q}{h-k}}$.

Factoring Irrational Expressions.

227. *Conversely*, if the quantity under the radical sign can be factored, we may express its root as the product of the roots of its factors. (Comp. 191, 199.)

EXERCISES.

Factor:

1. $(abc)^{\frac{1}{2}}$. Ans. $a^{\frac{1}{2}}b^{\frac{1}{2}}c^{\frac{1}{2}}$.
2. $(mnp)^{\frac{1}{3}}$.
3. $\sqrt{a^2 - ab}$. Ans. $\sqrt{a}\sqrt{a-b}$.
4. $\sqrt{m^2 - mx}$.
5. $\sqrt{a^2 - b^2}$.
6. $\sqrt{a^4 - b^4}$.
7. $\sqrt{a^4 - a^2b^2}$.
8. $\sqrt{55}$.
9. $\sqrt{66}$.
10. $(m^4 - 4m^2n^2)^{\frac{1}{2}}$.
11. $(a^2b - ab^2)^{\frac{1}{2}}$.
12. $(ax^3 + a^2x^2 + a^3x)^{\frac{1}{2}}$. Ans. $a^{\frac{1}{2}}x^{\frac{1}{2}}(x^2 + ax + a^2)^{\frac{1}{2}}$.
13. $(a^3m^2 - a^2m^3 + a^3m^3)^{\frac{1}{3}}$.
14. $(r^{2m} - x^{2n})^{\frac{1}{2}}$.

228. When the quantity under the radical sign is factored, one of the factors may be a perfect power. We may then extract the root of this power, and affix the surd root of the other factor to it.

In the problems of this and the next three articles the object is to remove as many factors as possible from under the radical sign.

EXERCISES.

Factor:

1. $\sqrt{a^2b}$. Ans. $\sqrt{a^2}\sqrt{b} = a\sqrt{b}$.
2. $\sqrt{mn^2}$.
3. $\sqrt{m^2n^3}$. Ans. $mn\sqrt{n}$.
4. $\sqrt{m^2n^3x^3}$.
5. $\sqrt{a^2bc^3}$.
6. $(g^2r^3s^4)^{\frac{1}{2}}$.
7. $(4a^2x)^{\frac{1}{2}}$.
8. $\sqrt{56}$. Ans. $2\sqrt{14}$.
9. $\sqrt{27}$.
10. $\sqrt{36a}$.
11. $\sqrt{2}\cdot\sqrt{72}$.
12. $\sqrt{x^2(a+b)}$.
13. $\sqrt{a^3b - a^2b^2}$.
14. $\sqrt{a^2x + 2abx + b^2x}$. Ans. $(a+b)\sqrt{x}$.

Here the quantity under the radical sign is equal to

$$(a^2 + 2ab + b^2)x = (a+b)^2x.$$

In questions of this class the beginner is apt to divide an expression like $\sqrt{a+b+c}$ into $\sqrt{a} + \sqrt{b} + \sqrt{c}$, which is wrong. The square root of the sum of several quantities cannot be reduced except by factoring. Hence such an expression as the square root of $a + b + c$ is not reducible, but must stand as it is.

15. $(m^3 - 2m^2n + mn^2)^{\frac{1}{2}}$.
16. $\sqrt{m^4 - 2m^3n + m^2n^2}$.
17. $\sqrt{4a^2y - 8acy + 4c^2y}$.
18. $\sqrt{a^4 - a^2b^2}$.
19. $\sqrt{a^2b^2 - b^4}$.
20. $\sqrt{4m^2x^2 - 16x^4}$.
21. $(m^2 + m^4)^{\frac{1}{2}}$.
22. $(n^2 + n^3)^{\frac{1}{2}}$.
23. $(a^3 - 2a^2 + a)^{\frac{1}{2}}$.
24. $(an^6 + 4an^5 + 4an^4)^{\frac{1}{2}}$.
25. $(n^5 + 4n^4 + 4n^3)^{\frac{1}{2}}$.

229. When a quantity to be factored by the preceding method is affected by a fractional exponent, this exponent may be divided into an integer (positive or negative) and a fraction, and the quantity factored accordingly.

EXAMPLES. $a^{\frac{5}{2}} = a^{2+\frac{1}{2}} = a^2 \,.\, a^{\frac{1}{2}}.$

$$a^{-\frac{1}{2}} = a^{-1+\frac{1}{2}} = a^{-1} \,.\, a^{\frac{1}{2}} = \frac{a^{\frac{1}{2}}}{a}.$$

EXERCISES.

Factor:

1. $m^{\frac{5}{2}}$.
2. $c^{\frac{3}{2}}m^{\frac{7}{2}}$.
3. $c^{\frac{5}{2}}m^{\frac{3}{2}}$.
4. $8^{\frac{3}{2}}$. Ans. $16\sqrt{2}$.
5. $24^{\frac{5}{2}}$.
6. $\sqrt{24a^3}$.
7. $ax^{\frac{1}{2}} + bx^{\frac{3}{2}} + cx^{\frac{5}{2}}$. Ans. $(a + bx + cx^2)x^{\frac{1}{2}}$.
8. $2m^{\frac{3}{2}} + 3nm^{\frac{5}{2}} - 4n^2m^{\frac{7}{2}}$.
9. $x^{-\frac{1}{2}} + x^{\frac{1}{2}}$. Ans. $\left(1 + \frac{1}{x}\right)x^{\frac{1}{2}}$.
10. $a^{-\frac{1}{2}}x^{\frac{1}{2}} - a^{\frac{1}{2}}x^{-\frac{1}{2}}$.
11. $a^{\frac{1}{2}} + 2a^{\frac{3}{2}} + 3a^{\frac{5}{2}}$.
12. $a^{\frac{1}{2}}x^{\frac{1}{2}} + a^{\frac{3}{2}}x^{\frac{3}{2}}$.
13. $8^{\frac{3}{2}}x^{\frac{1}{2}} + 8^{\frac{1}{2}}x^{\frac{3}{2}}$.

230. The preceding method may be applied to any other indicated roots as well as to square roots.

EXAMPLE 1. Factor $\sqrt[3]{16} = 16^{\frac{1}{3}}$.

We have $16^{\frac{1}{3}} = 8^{\frac{1}{3}} \times 2^{\frac{1}{3}} = 2 \,.\, 2^{\frac{1}{3}}$.

Ex. 2. Factor $\sqrt[3]{a^{-2}} + \sqrt[3]{a^4}$.

Solution. $a^{-\frac{2}{3}} + a^{\frac{4}{3}} = a^{-1} \,.\, a^{\frac{1}{3}} + aa^{\frac{1}{3}} = \left(\frac{1}{a} + a\right)a^{\frac{1}{3}}$.

EXERCISES.

Factor:

1. $24^{\frac{1}{2}}$. 2. $54^{\frac{2}{3}}$. 3. $32^{\frac{4}{3}}$. 4. $16^{\frac{5}{3}}$.

5. $c^{\frac{5}{3}} + c^{\frac{7}{3}}$. 6. $c^{\frac{5}{3}} + c^{\frac{8}{3}}$. 7. $\sqrt[3]{a^2b^3 - b^3}$.

231. The preceding system of removing factors from under the radical sign may be applied to fractions, applying the principles of §§ 200 and 228.

EXERCISES.

Factor:

1. $\sqrt{\frac{8}{27}}$. Ans. $\frac{\sqrt{8}}{\sqrt{27}} = \frac{2\sqrt{2}}{3\sqrt{3}} = \frac{2}{3}\sqrt{\frac{2}{3}}$.

2. $\sqrt{\frac{12}{125}}$. 3. $\frac{a^{\frac{3}{2}}}{b^{\frac{5}{2}}}$.

4. $\sqrt{\frac{c^2 + b^2}{a^2}}$. 5. $\sqrt{\frac{a^2 + a^4}{c^2 + c^4}}$.

6. $\left(\frac{a^2b^2 + a^2c^2}{b^4 + b^2c^2}\right)^{\frac{1}{2}}$. 7. $\left(\frac{a^2m + a^3}{c^3}\right)^{\frac{1}{2}}$.

8. $\frac{x^{\frac{1}{2}}}{a^{\frac{1}{2}}} + \frac{x^{\frac{3}{2}}}{a^{\frac{3}{2}}}$. 9. $\frac{mx^{\frac{1}{2}} - nx^{\frac{3}{2}}}{a^2y^{\frac{1}{2}}}$.

10. $\left(\frac{x^2 + x^3}{a^2 + a^3}\right)^{\frac{1}{2}}$.

Reduce the following fractions to their lowest terms:

11. $\frac{ax^{\frac{1}{2}} + cx}{ax^{\frac{1}{2}} - cx}$. Ans. $\frac{a + cx^{\frac{1}{2}}}{a - cx^{\frac{1}{2}}}$. 12. $\frac{my^{\frac{1}{2}} - ny^{\frac{3}{2}}}{my^{\frac{1}{2}} + ny^{\frac{3}{2}}}$.

13. $\frac{g + h}{a\sqrt{g + h}}$. 14. $\frac{\sqrt{3}}{\sqrt{6}}$.

15. $\frac{\sqrt{3}}{\sqrt{12}}$. 16. $\frac{gh + g^{\frac{1}{2}}h^{\frac{3}{2}}}{gh - g^{\frac{1}{2}}h^{\frac{3}{2}}}$.

17. $\frac{\sqrt{h^2 - m^2}}{h - m}$. Ans. $\frac{\sqrt{h + m}}{\sqrt{h - m}}$.

18. $\frac{\sqrt{a^2 - x^2}}{a + x}$. 19. $\frac{m + m^{\frac{1}{2}}}{am - bm^{\frac{1}{2}}}$.

20. $\dfrac{c+h+(c+h)^{\frac{1}{2}}}{c+h-(c+h)^{\frac{1}{2}}}$. Ans. $\dfrac{(c+h)^{\frac{1}{2}}+1}{(c+h)^{\frac{1}{2}}-1}$.

21. $\dfrac{a(1+n)+b(1+n)^{\frac{1}{2}}}{a(1+n)-b(1+n)^{\frac{1}{2}}}$. 22. $\dfrac{3+\sqrt{3}}{3-\sqrt{3}}$.

23. $\dfrac{\sqrt{a^4-2a^3+a^2}}{\sqrt{a^2x^2-2ax^2+x^2}}$. 24. $\dfrac{(a-x)^{\frac{1}{2}}}{(a^2-x^2)^{\frac{1}{2}}}$.

Multiplication of Surds of Different Degrees.

232. To multiply surds of different degrees they must first be reduced to surds of the same degree.

PROBLEM I. To reduce surds of different degrees to surds of the same degree.

RULE. *Express the indicated roots by fractional exponents, and reduce the exponents to their L.C.D.*

EXERCISES.

Reduce to surds of equal degree:

1. $\sqrt{a}$, $\sqrt{b^3}$, $\sqrt[3]{c}$. Ans. $a^{\frac{3}{6}}$, $b^{\frac{9}{6}}$, $c^{\frac{2}{6}}$.

2. $\sqrt{x}$, $\sqrt[3]{x}$, $\sqrt[4]{x}$. 3. $\sqrt[2]{m}$, $\sqrt[3]{m}$, $\sqrt[6]{m}$.

4. $\sqrt[3]{m^4}$. $\sqrt[2]{m^3}$, $\sqrt[3]{m}$. 5. $(a-b)^{\frac{1}{3}}$, $(a+b)^{\frac{1}{4}}$.

233. PROBLEM. II. To express a product of surds of different degrees as a simple surd.

RULE. *Reduce the surds to the same degree, and express the product by the fundamental theorems* (§§ 191, 202, 225).

EXAMPLE. Express as a single surd the product of the square root of a by the cube root of b.

$a^{\frac{1}{2}}b^{\frac{1}{3}} = a^{\frac{3}{6}}b^{\frac{2}{6}} = (a^3)^{\frac{1}{6}}(b^2)^{\frac{1}{6}}$ (§ 203) $= (a^3b^2)^{\frac{1}{6}}$ (§ 226) $= \sqrt[6]{a^3b^2}$.

The general principle applied in this case is this:

When we have to form a product of several factors affected with fractional exponents having a common denominator,

We may remove the denominator from all the exponents and write it outside the parenthesized product of the factors.

That is, we always have

$$a^{\frac{x}{n}}b^{\frac{y}{n}}c^{\frac{z}{n}} \ldots\ldots = (a^xb^yc^z \ldots\ldots)^{\frac{1}{n}}.$$

EXERCISES.

Express with single surds the products:

1. $\sqrt{2} \times \sqrt[3]{3}$. Ans. $72^{\frac{1}{6}}$.
2. $3^{\frac{1}{2}}2^{\frac{1}{3}}$.
3. $a^{\frac{1}{3}}b^{\frac{1}{4}}$.
4. $x^{\frac{1}{m}}y^{\frac{1}{n}}$.
5. $a^{\frac{1}{2}}b^{\frac{1}{3}}x^{\frac{1}{2}}$.
6. $2a^{\frac{1}{2}}$. Ans. $(4a)^{\frac{1}{2}}$.
7. $3b^{\frac{1}{2}}x^{\frac{2}{3}}$.
8. $32^{\frac{1}{2}}a^{\frac{3}{4}}b^{\frac{1}{4}}$.
9. $2^{\frac{1}{2}}(x-y)^{\frac{1}{3}}$.
10. $a^{\frac{1}{2}}(x+y)^{\frac{1}{3}}(x-y)^{\frac{1}{3}}$.
11. $2^{\frac{1}{2}}3^{\frac{1}{3}}$.
12. $3^{\frac{1}{3}}4^{\frac{1}{4}}5^{\frac{1}{2}}$.
13. $\dfrac{a^{\frac{1}{2}}}{c^{\frac{2}{3}}}$.
14. $\dfrac{(x+y)^{\frac{1}{n}}}{(x-y)^{\frac{2}{n}}}$.

234. *Corollary.* Because a rational quantity may be considered as affected with the exponent $\frac{1}{1}$, the preceding method can be applied to reduce the product of a rational quantity and a surd to the form of a surd.

EXAMPLE 1. Reduce $2b^{\frac{2}{3}}$ to a single index.

$$2b^{\frac{2}{3}} = 2^{\frac{3}{3}}b^{\frac{2}{3}} = (2^3b^2)^{\frac{1}{3}} = (8b^2)^{\frac{1}{3}} = \sqrt[3]{8b^2}.$$

Ex. 2. $3ab^{\frac{1}{2}}x^{\frac{1}{3}} = 3^{\frac{6}{6}}a^{\frac{6}{6}}b^{\frac{3}{6}}x^{\frac{2}{6}} = (3^6a^6b^3x^2)^{\frac{1}{6}} = \sqrt[6]{729a^6b^3x^2}$.

Ex. 3. $a + b\sqrt{m} = a + \sqrt{b^2m}$.

EXERCISES.

Reduce to single surds:

1. $2\sqrt{b}$.
2. $3\sqrt{x}$.
3. $5ax^{\frac{1}{2}}$.
4. $m\sqrt{n}$.
5. $(m+n)\sqrt{m-n}$.
6. $a\sqrt{a}$.
7. $(a+b)\sqrt{a+b}$.
8. $(m-n)(m-n)^{-\frac{1}{2}}$.
9. $\dfrac{m+n}{(m+n)^{\frac{2}{3}}}$.
10. $a(a+b)^{\frac{1}{3}}$.
11. $m(m-n)^{-\frac{2}{3}}$.
12. $\dfrac{ab^{\frac{1}{n}}c^{\frac{m}{n}}}{h}$.
13. $\dfrac{xy^2z^{\frac{1}{2}}}{p^2k^2m^{\frac{1}{2}}}$.
14. $\dfrac{x}{y} \cdot \dfrac{x^{\frac{1}{2}}}{y}$.
15. $\dfrac{ax^{\frac{1}{2}}y^{\frac{1}{3}}}{cp^{\frac{1}{2}}q^{\frac{2}{3}}}$.

235. Irrational expressions may be multiplied by multiplying each term of the multiplier into each term of the multiplicand and taking the sum of the products.

EXAMPLE. Develop the product $(a + b\sqrt{x})(g + h\sqrt{y})$.

$$\begin{array}{l} a + b\sqrt{x} \\ \underline{g + h\sqrt{y}} \\ ag + bg\sqrt{x} \\ \qquad\quad ah\sqrt{y} + bh\sqrt{xy} \\ \hline \end{array}$$

$$\text{Product} = ag + bg\sqrt{x} + ah\sqrt{y} + bh\sqrt{xy}.$$

EXERCISES.

Multiply:

1. $(a - c\sqrt{y})(a + b\sqrt{y})$.
2. $(a + \sqrt{y})(a - \sqrt{y})$.
3. $(a + n\sqrt{x})(b - m\sqrt{y})$.
4. $a^{\frac{1}{2}}(a^{\frac{1}{2}}b - a^{\frac{1}{2}}x)$.
5. $(am + n\sqrt{a - x})(n - m\sqrt{a - x})$.

   Ans. $mnx + (n^2 - am^2)\sqrt{a - x}$.
6. $(m - n\sqrt{y})(m + n\sqrt{y})$.
7. $(a + c\sqrt{x})(a - c\sqrt{y})$.
8. $(a^2 + c\sqrt{x^2 - a})(c + a\sqrt{x^2 - a})$.
9. $(a + \sqrt{a^2 - x^2})(a - \sqrt{a^2 - x^2})$.
10. $(m - \sqrt{m^2 - 2n^2})(m + \sqrt{m^2 - 2n^2})$.
11. $(m + \sqrt{m^2 - 1})(m - \sqrt{m^2 - 1})$.
12. $(\sqrt{a} + \sqrt{b})(\sqrt{a} - \sqrt{b})$.
13. $(c + \sqrt{x} + \sqrt{y})(c - \sqrt{x} - \sqrt{y})$.
14. $(a^{\frac{1}{2}} + b^{\frac{1}{2}} + c^{\frac{1}{2}})(a^{\frac{1}{2}} - b^{\frac{1}{2}} + c^{\frac{1}{2}})(a^{\frac{1}{2}} + b^{\frac{1}{2}} - c^{\frac{1}{2}})(a^{\frac{1}{2}} - b^{\frac{1}{2}} - c^{\frac{1}{2}})$.
15. $a^{\frac{1}{2}}[1 - (a - 1)^{\frac{1}{2}}][1 + 2(a - 1)^{\frac{1}{2}}]$.
16. $(\sqrt{x + a} + \sqrt{x - a})(\sqrt{x + a} - \sqrt{x - a})$.
17. $(m + \sqrt{x} + \sqrt{y})(m - \sqrt{x} + \sqrt{y})$.
18. $(x + y - \sqrt{x} - \sqrt{y})(1 + \sqrt{x} + \sqrt{y})$.
19. $\left(\dfrac{a^{\frac{1}{2}}}{c^{\frac{1}{2}}} + \dfrac{c^{\frac{1}{2}}}{a^{\frac{1}{2}}}\right)\left(\dfrac{a^{\frac{1}{2}}}{c^{\frac{1}{2}}} - \dfrac{c^{\frac{1}{2}}}{a^{\frac{1}{2}}}\right)$.

20. $\left(\dfrac{\sqrt{x+a}}{\sqrt{x-a}}+1\right)\left(\dfrac{\sqrt{x+a}}{\sqrt{x-a}}-1\right)$.

21. $[(m^2+1)^{\frac{1}{2}}+m][(m^2+1)^{\frac{1}{2}}-m]$.

22. $\left[\sqrt{\left(1+\dfrac{1}{a^2}\right)}+\dfrac{1}{a}\right]\left[\sqrt{\left(1+\dfrac{1}{a^2}\right)}-\dfrac{1}{a}\right]$.

236. *Rationalizing Fractions.* The quotient of two surds may be expressed as a fraction with a rational numerator or a rational denominator by multiplying both terms by the proper multiplier.

EXAMPLE. By multiplying both terms of $\dfrac{\sqrt{6}}{\sqrt{7}}$ by $\sqrt{6}$, we have

$$\frac{\sqrt{6}}{\sqrt{7}}=\frac{6}{\sqrt{42}},$$

and the numerator is rational.

In the same way, multiplying by $\sqrt{7}$, we have

$$\frac{\sqrt{6}}{\sqrt{7}}=\frac{\sqrt{42}}{7}.$$

EXERCISES.

Express the following fractions with rational denominators:

1. $\dfrac{\sqrt{3}}{\sqrt{5}}$.

2. $\dfrac{\sqrt{5}}{2\sqrt{2}}$.

3. $\dfrac{\sqrt{18}}{\sqrt{3}}$.

4. $\dfrac{\sqrt{a+b}}{\sqrt{a-b}}$.

5. $\dfrac{\sqrt{m}}{\sqrt{n}}$.

6. $\dfrac{\sqrt{x^2-1}}{\sqrt{x^2+1}}$.

7. $\sqrt{\left(\dfrac{m+n}{m-n}\right)}$.

8. $\dfrac{a^{\frac{2}{3}}}{c^{\frac{1}{3}}}$. Ans. $\dfrac{a^{\frac{2}{3}}c^{\frac{2}{3}}}{c}$.

9. $\dfrac{m^{\frac{1}{3}}}{n^{\frac{2}{3}}}$.

10. $\dfrac{1}{a^{\frac{2}{3}}}$.

11. $\dfrac{(a+x)^{\frac{1}{3}}}{(a-x)^{\frac{2}{3}}}$.

237. If one term of the fraction contains a quadratic surd, it may be rationalized by multiplying by the term itself with the sign of the surd changed.

EXAMPLE. To rationalize the denominator in

$$\frac{\sqrt{R}}{P - Q\sqrt{R}},$$

we multiply both terms by $P + Q\sqrt{R}$. The numerator becomes

$$QR + P\sqrt{R}.$$

The denominator becomes

$$(P - Q\sqrt{R})(P + Q\sqrt{R}) = P^2 - Q^2R.$$

So we have

$$\frac{\sqrt{R}}{P - Q\sqrt{R}} = \frac{QR + P\sqrt{R}}{P^2 - Q^2R}.$$

EXERCISES.

Reduce the following fractions to others with rational denominators:

1. $\dfrac{a + b\sqrt{x}}{a - b\sqrt{x}}$.

2. $\dfrac{a - c\sqrt{y}}{a + c\sqrt{y}}$.

3. $\dfrac{\sqrt{b}}{a - \sqrt{b}}$.

4. $\dfrac{\sqrt{x}}{a + c\sqrt{x}}$.

5. $\dfrac{\sqrt{m + n}}{a - \sqrt{m - n}}$.

6. $\dfrac{a + x^{\frac{3}{2}}}{a - x^{\frac{3}{2}}}$.

7. $\dfrac{\sqrt{A} + \sqrt{B}}{\sqrt{A} - \sqrt{B}}$.

8. $\dfrac{\sqrt{m} + \sqrt{(m + n)}}{\sqrt{m} - \sqrt{(m + n)}}$.

9. $\dfrac{\sqrt{x} - \sqrt{(a - x)}}{\sqrt{x} + \sqrt{(a - x)}}$.

10. $\dfrac{1}{m + \sqrt{m^2 - a^2}}$.

11. $\dfrac{1}{m - \sqrt{m^2 - a^2}}$.

12. $\dfrac{(m + n)^{\frac{1}{2}} - (m - n)^{\frac{1}{2}}}{(m + n)^{\frac{1}{2}} + (m - n)^{\frac{1}{2}}}$.

13. $\dfrac{x - y}{x + y - 2\sqrt{xy}}$.

14. $\dfrac{1}{x + x^{\frac{1}{2}}}$.

Irrational Factors.

238. By introducing surds many expressions can be factored which are prime when only rational factors are considered. The following theorem may be applied for this purpose:

THEOREM. *The difference of any two quantities is equal to the product of the sum and difference of their square roots.*

Proof. If a and b are the quantities, we shall have

$$a - b = (a^{\frac{1}{2}} - b^{\frac{1}{2}})(a^{\frac{1}{2}} + b^{\frac{1}{2}}),$$

as can be proved by multiplying, or by § 123.

EXERCISES.

Factor:

1. $a - x$. 2. $a - 1$. 3. $a^2 - bx$.

4. $16 - 3$. 5. $\frac{1}{a} - \frac{1}{c}$. 6. $x^2 - (a + b)$.

7. $(x - c)^2 - \frac{x}{4}$. 8. $x^2 - (p + q)$.

9. $(x - c)^2 - \frac{1}{4}(p - q)$. 10. $\frac{x}{a} - \frac{y}{c}$.

11. $\frac{x^2}{a^2} - (a + b)$. 12. $\frac{m^2 - n^2}{m - n} - \frac{m - n}{m + n}$.

13. $x^2 - 4ax + 4a^2 - b$. 14. $x^2 - 2x - 5$.

15. $x^2 - 2ax + a^2 - (p + q)$. 16. $2a^2 - 4(p - q)$.

239. *Irrational Square Roots.* When a trinomial consists of two positive terms with twice the square root of their product, its square root may be found by the method of § 131.

EXERCISES.

Find the irrational square roots of the following expressions:

NOTE. A few of the roots are rational.

1. $a - 2\sqrt{ab} + b$. Ans. $\sqrt{a} - \sqrt{b}$.

2. $a + 2\sqrt{ab} + b$.

3. $a + \frac{1}{a} + 2$. Ans. $a^{\frac{1}{2}} + \frac{1}{a^{\frac{1}{2}}}$.

4. $a^2 - 2ax^{\frac{1}{2}} + x.$

5. $a^2 - 2 + \dfrac{1}{a^2}.$

6. $a^3 - 2 + \dfrac{1}{a^3}.$

7. $a^n - 2 + \dfrac{1}{a^n}.$

8. $9 + 5 - 6\sqrt{5}.$

9. $\dfrac{1}{a} + \dfrac{1}{b} - \dfrac{2}{\sqrt{ab}}.$

10. $\dfrac{1}{4} + \dfrac{1}{3} + \dfrac{1}{9}.$

11. $\dfrac{1}{16} + \dfrac{1}{25} + \dfrac{1}{10}.$

12. $\dfrac{m}{16} - \dfrac{m}{10} + \dfrac{m}{25}.$

13. $\dfrac{a}{c} - 2 + \dfrac{c}{a}.$

14. $\dfrac{a}{c} - 4 + \dfrac{4c}{a}.$

15. $\dfrac{m}{n} - 6 + \dfrac{9n}{m}.$

16. $x + y - 2 + \dfrac{1}{x+y}.$

17. $(x+y) + (x-y) + 2\sqrt{x^2 - y^2}.$

18. $x + 2x^{\frac{1}{2}} + 1.$

19. $a^{\frac{1}{2}} - 2a^{\frac{1}{4}} + 1.$

20. $a^{\frac{3}{2}} - 2a + a^{\frac{1}{2}}.$

21. $\dfrac{m^{\frac{1}{2}}}{4} - \dfrac{m^{\frac{1}{4}}x^{\frac{1}{4}}}{2} + \dfrac{x^{\frac{1}{2}}}{4}.$

22. $x^{\frac{5}{3}} + 2x + x^{\frac{1}{3}}.$

23. $x^{\frac{4}{5}} - x + \dfrac{1}{4}x^{\frac{6}{5}}.$

24. $m^{\frac{1}{2}} - 2 + m^{-\frac{1}{2}}.$

25. $(x+a) + 2(x+a)^{\frac{1}{2}} + 1.$

26. $4(x-a) + 4m(x-a)^{\frac{1}{2}} + m^2.$

27. $(x-a) - 2a^{\frac{1}{2}}(x-a)^{\frac{1}{2}} + a.$

28. $1 - \dfrac{x}{c} + 2c\sqrt{1 - \dfrac{x}{c}} + c^2.$

29. $a^2b^2 + 4 + 4a^{-2}b^{-2}.$

240. *Square Roots of Irrational Binomials.* If we have such a binomial as $a + \sqrt{b}$, its square root may be expressed in the form

$$\sqrt{a + \sqrt{b}},$$

in which a surd is itself under the radical sign.

If this root can be so reduced that no surd shall be under the radical sign, it is said to be **reducible**; otherwise it is **irreducible**.

PROBLEM. To find whether the square root of an irrational binomial is reducible, and when it is to express it in the reduced form.

Solution. Let us suppose

$$\sqrt{a+\sqrt{b}} = \sqrt{x}+\sqrt{y}.$$

Squaring, $$a+\sqrt{b} = x+y+2\sqrt{xy}.$$

This equation may be satisfied by putting

$$x+y=a,$$

$$xy=\frac{b}{4}.$$

By putting $\sqrt{a-\sqrt{b}} = \sqrt{x}-\sqrt{y}$, we get the same expressions for x and y.

Therefore, *if we can find two rational quantities, x and y, such that their sum shall be equal to the rational term of the binomial and their product to one fourth the square of the surd term,*

Then the root is the sum or difference of the square roots of these quantities.

EXAMPLE 1. $\sqrt{5+2\sqrt{6}} = \sqrt{5+\sqrt{24}}.$

We have to find two numbers of which the sum shall be 5 and the product $24 \div 4 = 6$. Such numbers are 2 and 3. Therefore

$$\sqrt{5+2\sqrt{6}} = \sqrt{3}+\sqrt{2},$$

and $$\sqrt{5-2\sqrt{6}} = \sqrt{3}-\sqrt{2}.$$

REMARK. We see that when we express the binomial in the form $a \pm 2\sqrt{c}$, a will be the sum and c the product of the required numbers

EXERCISES.

Express the square roots of:

1. $7+2\sqrt{12}$.
2. $7-4\sqrt{3}$.
3. $7-2\sqrt{6}$.
4. $9-\sqrt{80}$.
5. $3+\sqrt{8}$.
6. $6+4\sqrt{2}$.
7. $6+\sqrt{20}$.
8. $7+\sqrt{40}$.
9. $8+\sqrt{60}$.
10. $14-8\sqrt{3}$.

11. $15 - 6\sqrt{6}$.
12. $17 + 4\sqrt{15}$.
13. $2m + 2\sqrt{m^2 - 1}$.
14. $m + \sqrt{m^2 - 1}$.
15. $2 - 2\sqrt{1 - m^2}$.
16. $1 - \sqrt{1 - m^2}$.
17. $2a + 2\sqrt{a^2 - b^2}$.
18. $a - \sqrt{a^2 - b^2}$.
19. $b^2 + 2a\sqrt{b^2 - a^2}$.
20. $4a + 2\sqrt{4a^2 - b^2}$.
21. $2a - b - 2\sqrt{a^2 - ab}$.
22. $4x - 2\sqrt{4x^2 - a^2}$.

To Complete the Square.

241. If one term of a binomial is a perfect square, such a term can always be added to the binomial that the trinomial thus formed shall be a perfect square.

This operation is called **completing the square.**

Proof. Call a the square root of the term which is a perfect square, which term we suppose the first, and call m the other term, so that the given binomial shall be

$$a^2 + m.$$

Add to this binomial the term $\frac{m^2}{4a^2}$, and it will become

$$a^2 + m + \frac{m^2}{4a^2}.$$

This is a perfect square, namely, the square of

$$a + \frac{m}{2a};$$

that is,
$$a^2 + m + \frac{m^2}{4a^2} = \left(a + \frac{m}{2a}\right)^2.$$

Hence the following

RULE. *Add to the binomial the square of the second term divided by four times the first term.*

EXAMPLE. What term must be added to the expression

$$x^2 - 4ax$$

to make it a perfect square?

The rule gives for the term to be added

$$\frac{(-4ax)^2}{4x^2} = 4a^2.$$

Therefore the required perfect square is

$$x^2 - 4ax + 4a^2 = (x - 2a)^2.$$

EXERCISES.

Complete the square in each of the following expressions, and extract the root of the completed square:

1. $a^2 - 2ab$.
2. $a^2 + 4ax$.
3. $4a^2 - 8ax$.
4. $4a^4 + 4a^3x$.
5. $a^2 + b$.
6. $a^2 - b$.
7. $a^4 - 4a^3$.
8. $a^2x^2 + a^3x$.

242. In the preceding article, by adding $4a^2$ to the binomial $x^2 - 4ax$, we formed the equation

$$x^2 - 4ax + 4a^2 = (x - 2a)^2.$$

By transposing the added term, we have

$$x^2 - 4ax = (x - 2a)^2 - 4a^2.$$

The original binomial is now expressed as the difference of two squares. Therefore the above process is a solution of the problem

Having a binomial of which one term is a perfect square, to express it as a difference of two squares.

EXERCISES.

Express the following binomials as differences of two squares:

1. $a^2 - 2ab$. Ans. $(a - b)^2 - b^2$.
2. $a^2 - 4ab$.
3. $x^2 + ax$.
4. $x^2 + 2ax$.
5. $a^2x^2 - a^3x$.
6. $\frac{x^2}{a^2} - \frac{2x}{a}$.
7. $\frac{x^4}{a^2} + x^2$.
8. $4a^2x^2 - 4b^3x$.
9. $m^2x^2 - 1$.
10. $m^2x^2 + 2$.
11. $\frac{1}{a^2} + 2$.
12. $\frac{1}{a^2x^2} - 2$.
13. $a^2x^2 - 8a^3x$.
14. $\frac{x^2}{4a^2} + \frac{x}{a}$.
15. $c^2 + \frac{1}{c^2}$.

MEMORANDA FOR REVIEW.

Powers and Roots of Monomials.

Define: Power; Square; Cube; nth Power; Index of power; nth Root; Index of root; Square root; Cube root; Evolution.

Involution.
- Of products; Theorem; Proof.
- Of fractions; Theorem; Proof.
- Of powers; Theorem; Proof.
- Algebraic sign of powers; Theorem.

Evolution.
- Sign of evolution; Index; Vinculum.
- Division of exponents; Theorem.
- Sign of even root.
- Of products; Rule; Proof.
- Of fractions; Rule; Proof.

Fractional Exponents.
- Indicate root; Explain.
- Theorem of power and root (§ 202).
- Significance of terms of the fractional exponent; Explain.
- Powers / Roots of expressions having fractional exponents; How formed.

Negative Exponents.
- Meaning; Explain application of preceding rules to negative exponents.

Powers and Roots of Polynomials.

Binomial Theorem.
- Powers of $1 + x$; How formed.
- Expression for coefficients.
- Define binomial coefficients.
- When one term is negative.
- Powers of $a + b$.

Square Root of Polynomial or Number.
- Arrangement of terms.
- Criterion that a root is possible.
- Rule for polynomial; Explain.
- When exact root cannot be extracted.
- Square root of numbers; Rule; Explain.

Irrational Expressions.

Define: Rational; Irrational; Perfect power; Irreducible; Surd; Quadratic surd; Similar surds.

Aggregation of Similar surds; Rule.

Surd Factors and Products.
- Theorem of roots; Proof.
- Products / Factors — of surds of same degree.
- When one factor is a perfect power; Rule.
- Case of fractions.
- Reduction of surds to common degree; Rule.
- Product of surds of same degree; Rule.
- Of irrational polynomials.

Rationalizing Irrational fractions; Rule.

Irrational Factors and Roots.
- To factor any binomial; Theorem.
- Irrational square roots.
- Square roots of binomial surds; When reducible; Expression for root.

To complete the square; Rule; Result.

CHAPTER V.

QUADRATIC EQUATIONS.

SECTION I. PURE QUADRATIC AND OTHER EQUATIONS.

243. *Def.* A **quadratic equation** is one which, when cleared of fractions, contains the second and no higher power of the unknown quantity.

REMARK. A quadratic equation is also called an equation of the *second degree.*

Def. A **pure** quadratic equation is one which contains no power of the unknown quantity except the second.

Def. A **complete** quadratic equation is one which contains both the first and second powers of the unknown quantity.

Pure Quadratic Equations.

244. If, on clearing of fractions, arranging according to powers of x and transposing, we put

$$A \equiv \text{the coefficient of } x^2,$$
$$B \equiv \text{the terms not containing } x,$$

a pure quadratic equation will reduce to

$$Ax^2 = B.$$

Dividing by A, we have

$$x^2 = \frac{B}{A}.$$

Extracting the square root of both members,

$$x = \pm\sqrt{\frac{B}{A}} = \pm\frac{B^{\frac{1}{2}}}{A^{\frac{1}{2}}}.$$

245. *Positive and Negative Roots.* Since the square root of a quantity may be either positive or negative, it follows that when we have an equation such as

$$x^2 = a$$

and extract the square root, we may have either

$$x = +a^{\frac{1}{2}}$$

or $$x = -a^{\frac{1}{2}}.$$

Hence there are two roots to every such equation, the one positive and the other negative. We express this pair of roots by writing

$$x = \pm a^{\frac{1}{2}},$$

the expression $\pm a^{\frac{1}{2}}$ meaning either $+a^{\frac{1}{2}}$ or $-a^{\frac{1}{2}}$.

REMARK. It might seem that since the square root of x^2 is either $+x$ or $-x$, we should write

$$\pm x = \pm a^{\frac{1}{2}},$$

having the four equations
$$\begin{aligned} x &= a^{\frac{1}{2}}, \\ x &= -a^{\frac{1}{2}}, \\ -x &= +a^{\frac{1}{2}}, \\ -x &= -a^{\frac{1}{2}}. \end{aligned}$$

But the first and fourth of these equations give identical values of x by simply changing their signs, and so do the second and third; hence more than two of the equations are unnecessary.

EXAMPLE. Solve the equation

$$\frac{x - na}{x - a} = \frac{nx - b}{x - b}.$$

Clearing of fractions, we have

$$(x - b)(x - na) = (x - a)(nx - b),$$

or $$x^2 - bx - nax + nab = nx^2 - nax - bx + ab.$$

Removing equal terms,

$$x^2 + nab = nx^2 + ab.$$

Transposing, $$nx^2 - x^2 = nab - ab,$$

or $$(n - 1)x^2 = (n - 1)ab.$$

Hence $$x^2 = ab.$$

Extracting root, $$x = \pm \sqrt{ab}.$$

246. Studying the preceding process, we see that a pure equation is solved by the following

RULE. 1. *Clear the equation of fractions.*

2. *Transpose all terms containing x^2 as a factor to one member; those not containing x to the other.*

3. *Divide by the aggregated coefficient of the square of the unknown quantity.*

4. *Extract the square root of both members.*

EXERCISES.

Solve the following equations, regarding x, y or z as the unknown quantity:

1. $x^2 = a.$
2. $x^2 = c^2.$
3. $3x^2 = 27.$
4. $2x^2 - 98 = 0.$
5. $3x^2 - 36 = 0.$
6. $7x^2 - 4 = 0.$
7. $(a - b)x^2 = c.$
8. $(a - b)x^2 = a^2 - b^2.$
9. $mx^2 - m^2 + nx^2 + n^2 = 0.$
10. $(x + 5)(x - 5) = 11.$
11. $(x - a^2)(x - c^2) + (a^2 + c^2)x = 3a^2c^2.$
12. $ax^2 - b + c = 0.$
13. $(x - a)(x - b) + (x + a)(x + b) = a^2.$
14. $x = \frac{a}{x} + \frac{a^2 - x^2}{x}.$
15. $(x^2 + a^2)^{\frac{1}{2}} = \frac{b^2}{(x^2 + a^2)^{\frac{1}{2}}}.$
16. $\frac{10}{x^2 + 2} = \frac{8}{5x^2 - 32}.$
17. $x + \frac{1}{x} = ax.$
18. $mx + \frac{m}{x} = ax.$
19. $x(a - x) - x(a + x) = 0.$
20. $\left(\frac{a}{x - a} - \frac{c}{x - c}\right)x = a^2c^2.$
21. $\frac{x}{m} - \frac{m}{x} = mx.$
22. $a : x = x : c.$
23. $a : bx = cx : a.$
24. $x^2 + a : x^2 - a = p + q : p - q.$
25. $(x + a + b)(x - a + b) + (x + a - b)(x - a - b) = 0.$
26. $\frac{x - a}{x + 1} + \frac{x + a}{x - 1} = 2c.$
27. $x + a + 2b : x + a - 2b = b - 2a + 2x : b + 2a - 2x.$

Reduce by composition and division (§ 185).

Pure Equations of any Degree.

247. *Def.* An equation which, when cleared of fractions, contains only the nth power of the unknown quantity is called a **pure equation of the nth degree.**

A pure equation of the third degree is called a pure **cubic equation.**

One of the fourth degree is called a pure **biquadratic equation.**

Def. A pure equation reduced to the form

$$x^n = a \tag{1}$$

is called a **binomial equation.**

248. *Solution of a Binomial Equation.*

1. *When the exponent is a whole number.* If we extract the nth root of both members of the equation (1), these roots will, by Axiom V., still be equal. The nth root of x^n being x, and that of a being $a^{\frac{1}{n}}$, we have

$$x = a^{\frac{1}{n}},$$

and the equation is solved.

2. *When the exponent is fractional.* Let the equation be

$$x^{\frac{m}{n}} = a.$$

Raising both members to the nth power, we have

$$x^m = a^n.$$

Extracting the mth root,

$$x = a^{\frac{n}{m}}.$$

If the numerator of the exponent is unity, the equation will take the form

$$x^{\frac{1}{n}} = a.$$

By raising it to the nth power,

$$x = a^n.$$

Hence the binomial equation always admits of solution by forming powers, extracting roots, or both.

EXERCISES.

Find the values of x in the following equations:

1. $\dfrac{a}{x^3} = a.$

2. $\dfrac{a}{x^4} = a.$

3. $\dfrac{a}{x^{\frac{1}{2}}} = b.$

4. $\dfrac{a}{x^{\frac{1}{2}} - c} = \dfrac{c}{x^{\frac{1}{2}} - a}.$

5. $\dfrac{8}{x} = \dfrac{x^2}{27}.$

6. $\dfrac{m}{x^{\frac{1}{2}}} = m^{\frac{1}{2}}.$

7. $\dfrac{a + b}{x^3} = \dfrac{a^2 - b^2}{x}.$

8. $\dfrac{x^{\frac{1}{3}}}{a} = \dfrac{m}{x^{\frac{1}{2}}}.$

9. $\dfrac{x^{\frac{1}{2}}}{c^{\frac{1}{3}}} = \dfrac{e^{\frac{2}{3}}}{x^{\frac{1}{6}}}.$

10. $\sqrt{(x^2 - a^2)} = c.$

11. $\sqrt{mx^2 - h^2} = x.$

   Square both members.

12. $(x - h)^{\frac{1}{3}} = m^{\frac{2}{3}}.$

13. $(x^{\frac{1}{2}} - c^{\frac{1}{2}})^{\frac{1}{3}} = q^{\frac{1}{6}}.$

14. $(x^2 + h^2)^{\frac{1}{2}} = \dfrac{a^2}{(x^2 - h^2)^{\frac{1}{2}}}.$

249. *Problems leading to Binomial Equations.*

1. Find two numbers one of which is three times the other, and the sum of whose squares is 90.

2. Find two numbers of which one is twice the other, and of which the product is 48.

3. Two numbers are required whose ratio is 2 : 3 and the difference of whose squares is $61\frac{1}{4}$.

   Note § 188, *Cor.*

4. Two numbers are required whose ratio is 3 : 5 and the difference of whose cubes is 2646.

5. Find three numbers of which the second is twice the first, the third twice the second, and the sum of the squares 378.

6. Find a number such that if 4 be added to it and subtracted from it the product of the sum and difference shall be 273.

7. Of what two numbers is one twice the other, and the difference of the squares 60?

8. Find two numbers whose ratio is 2 : 3 and the sum of whose cubes is 945.

9. Find two numbers whose ratio is 3 : 4 and the square of whose sum is 392.

10. What number multiplied by its own square produces 1331?

11. What number multiplied by its own square root produces 729?

12. Find two numbers in the ratio 2 : 5 the difference of whose squares is greater than the square of their difference by 216.

13. Find two numbers in the ratio $m : n$ whose product is a^2.

14. Find two numbers in the ratio $m : n$ the difference of whose squares is greater by $m^2 - n^2$ than the square of their difference.

15. What number multiplied by its own square root produces $48\sqrt{6}$?

16. Of what number is the product of the square and cube roots 243?

17. Find three numbers in the ratio 1 : 2 : 3 the sum of whose squares shall be 350.

SECTION II. COMPLETE QUADRATIC EQUATIONS.

250. *Normal Form.* Every complete quadratic equation can, by clearing of fractions and transposing, be reduced to the form

$$ax^2 + bx + c = 0,$$

in which a, b and c may represent any numbers or algebraic expressions which do not contain the unknown quantity x.

Def. The form

$$ax^2 + bx + c = 0 \qquad (1)$$

is called the **normal form** of the quadratic equation.

The quantities a, b and c are called **coefficients** of the equation.

251. *General Form.* If we divide all the terms by a, the equation (1) will become

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0.$$

Now let us put, for brevity,

$$p \equiv \frac{b}{a}; \qquad q \equiv \frac{c}{a}.$$

The last equation will then be written

$$x^2 + px + q = 0. \tag{2}$$

Def. The form

$$x^2 + px + q = 0$$

is called the **general form** of a quadratic equation, because it is a form to which every such equation may be reduced.

Solution of a Complete Quadratic Equation.

252. *The Equation in its General Form.*

We first transpose q, obtaining from the general equation (2)

$$x^2 + px = -q.$$

By § 241 the first member may be made a perfect square by adding $\frac{p^2}{4}$.

Adding this quantity to both members, we have

$$x^2 + px + \frac{p^2}{4} = \frac{p^2}{4} - q.$$

The first member of the equation is now a perfect square. Extracting the square roots of both members, we have

$$x + \frac{p}{2} = \pm\sqrt{\frac{p^2}{4} - q}.$$

Transposing $\frac{p}{2}$, we obtain a value of x which may be put in either of the several forms,

$$x = -\frac{p}{2} \pm \sqrt{\frac{p^2}{4} - q},$$

$$x = -\frac{p}{2} \pm \frac{\sqrt{p^2 - 4q}}{2},$$

or

$$x = \tfrac{1}{2}(-p \pm \sqrt{p^2 - 4q}),$$

and the equation is solved.

253. *The Two Roots.* Since the square root in the expression for x may be either positive or negative, there will be two roots to every quadratic equation, the one formed from the positive and the other from the negative surd. If we distinguish these roots as x_1 and x_2, their values will be

$$\left.\begin{aligned} x_1 &= \frac{-p+\sqrt{(p^2-4q)}}{2}, \\ x_2 &= \frac{-p-\sqrt{(p^2-4q)}}{2}. \end{aligned}\right\} \qquad (3)$$

EXAMPLE 1. Solve the equation

$$\frac{x}{x-3} - \frac{2x}{x+3} = -2.$$

Clearing of fractions,

$$\begin{aligned} x^2+3x-2x^2+6x &= -2(x^2-9) \\ &= -2x^2+18. \end{aligned}$$

Transposing and arranging,

$$x^2+9x = 18.$$

Completing the square,

$$x^2+9x+\frac{81}{4} = 18+\frac{81}{4} = \frac{153}{4}.$$

Extracting the root, $\qquad x+\frac{9}{2} = \pm\frac{\sqrt{153}}{2};$

whence $\qquad x = \frac{-9\pm\sqrt{153}}{2}.$

Ex. 2. $\qquad -x^2+3x-1=0.$

The coefficient of x^2 being negative, we must, in order to form a perfect square, change the sign of each term. The equation then becomes

$$x^2-3x+1=0,$$

or

$$x^2-3x \quad = -1.$$

Completing square,

$$x^2-3x+\frac{9}{4} = \frac{9}{4}-1 = \frac{5}{4};$$

Extracting root, $\qquad x-\frac{3}{2} = \pm\frac{\sqrt{5}}{2};$

whence $\qquad x = \frac{3\pm\sqrt{5}}{2}.$

EXERCISES.

Solve:

1. $x^2 - 4x = 3$.
2. $x^2 - 4x = 0$.
3. $\dfrac{x-2}{x+2} = \dfrac{2x-2}{x+17}$.
4. $\dfrac{x-2}{x+2} = \dfrac{x+1}{2x-8}$.
5. $x^2 - 2x = 3$.
6. $x^2 - 5x = 14$.
7. $2(x-2)(x-3) = (x-1)(x+2) - 16$.
8. $(2x+2)(x+3) - (x-1)(x+2) = 5$.
9. $x^2 + ax + b = 0$.
10. $x^2 + ax - b = 0$.
11. $x^2 - ax + b = 0$.
12. $x^2 - ax - b = 0$.

254. *The Equation in its Normal Form.* If in the equation (1),

$$ax^2 + bx + c = 0,$$

we transpose c and multiply the equation by a, we obtain the equation

$$a^2x^2 + abx = -ac.$$

To make the first member a perfect square, we add $\dfrac{b^2}{4}$ to each member (§ 241), giving

$$a^2x^2 + abx + \frac{b^2}{4} = \frac{b^2}{4} - ac.$$

Extracting the square root of both members, we have

$$ax + \frac{b}{2} = \frac{1}{2}\sqrt{(b^2 - 4ac)}.$$

We then obtain, by transposing and dividing,

$$x = -\frac{b}{2a} \pm \frac{1}{2a}\sqrt{(b^2 - 4ac)}$$

$$= \frac{-b \pm \sqrt{(b^2 - 4ac)}}{2a}.$$

Hence the two roots are

$$\left.\begin{aligned} x_1 &= \frac{-b + (b^2 - 4ac)^{\frac{1}{2}}}{2a} \\ \text{and} \qquad x_2 &= \frac{-b - (b^2 - 4ac)^{\frac{1}{2}}}{2a}. \end{aligned}\right\} \qquad (4)$$

We can always find the roots of a given quadratic equation by substituting the coefficients in the preceding expression for x. But the student is advised to solve each separate equation by the process just given, which is embodied in the following rule.

255. RULE. 1. *Reduce the equation to its normal or its general form, as may be most convenient.*

2. *Transpose the terms which do not contain x to the second member.*

3. *If the coefficient of x^2 is unity, add one fourth the square of the coefficient of x to both members of the equation and extract the square root.*

4. *If the coefficient of x^2 is not unity, either divide by it so as to reduce it to unity, or multiply or divide all the terms by such a factor or divisor that the term in x^2 shall become a perfect square.*

5. *But if the term in x^2 is already a perfect square, no multiplication or division need be performed.*

6. *Complete the square by the rule of § 241, and extract the square root.*

EXAMPLE. Solve the equation

$$\frac{x-1}{x-4} = 2x.$$

Clearing of fractions and transposing, we find the equation to become

$$2x^2 - 9x + 1 = 0.$$

$\times$ 2, $$4x^2 - 18x = -2.$$

Adding $\frac{9^2}{4} = \frac{81}{4}$ to each member of the equation, we have

$$4x^2 - 18x + \frac{9^2}{4} = \frac{81}{4} - 2 = \frac{73}{4}.$$

Extracting the square root,

$$2x - \frac{9}{2} = \sqrt{\frac{73}{4}} = \frac{\sqrt{73}}{2};$$

whence we find $$x = \frac{9 \pm \sqrt{73}}{4}.$$

So the two roots are

$$x_1 = \tfrac{1}{4}(9 + \sqrt{73}),$$

$$x_2 = \tfrac{1}{4}(9 - \sqrt{73}).$$

EXERCISES.

Solve the equations:

1. $x^2 + hx = \frac{1}{4}k^2$.

Steps of Solution.

$$x^2 + hx + \frac{h^2}{4} = \frac{h^2}{4} + \frac{1}{4}k^2 = \frac{h^2 + k^2}{4},$$

$$x + \frac{h}{2} = \frac{(h^2 + k^2)^{\frac{1}{2}}}{2},$$

$$x = \frac{-h \pm (h^2 + k^2)^{\frac{1}{2}}}{2};$$

or

$$x_1 = \frac{-h + (h^2 + k^2)^{\frac{1}{2}}}{2},$$

$$x_2 = \frac{-h - (h^2 + k^2)^{\frac{1}{2}}}{2}$$

2. $x^2 + 2hx = k^2$.

3. $x^2 + 2hx + k^2 = 0$.

4. $x^2 + 4mx - n = 0$.

5. $x^2 - 2ax = 3a^2$.

6. $x^2 - 4ax + 4a^2 = 0$.

7. $m^2x^2 + 2nx = p$.

8. $mx^2 - m^2x = 4m^3$.

9. $ax^2 - 2bx = 4b^2$.

10. $\frac{x^2}{a^2} - 2\frac{x}{a} - 3 = 0$.

11. $\frac{x^2}{2a^2} + 6\frac{x}{a} = 3m$.

12. $\frac{x^2}{4a^2} - mx = n$.

13. $m^2x^2 - 2m^3x + m^4 = 1$.

14. $4cx^2 - (a + b)x + c = 0$.

15. $(a + b)x^2 + (a - b)x = a^2 - b^2$.

16. $\frac{x - a}{x - c} + \frac{x - c}{x - a} = \frac{5}{2}$.

17. $\dfrac{m - \dfrac{x - a}{x + a}}{m + \dfrac{x + a}{x - a}} = -1$.

18. $\frac{1}{x + 5} = \frac{1}{2} + \frac{1}{3} + \frac{1}{x}$.

19. $\frac{x}{3} - \frac{3}{x} = \frac{x}{8} + \frac{8}{x}$.

20. $\frac{x}{m} - \frac{m}{x} = \frac{x}{n} + \frac{n}{x}$.

21. $\frac{1}{x} + \frac{1}{x + a} = \frac{1}{a} + \frac{1}{2a}$.

22. $\frac{4}{x + 2} + \frac{5}{x + 4} = \frac{10}{x + 6}$.

23. $\frac{x + 1}{x + 2} + \frac{x - 1}{x - 2} = \frac{4x - 1}{2x - 1}$.

24. $a^2 + b^2 + x^2 = 1 - 2abx$.

25. $(a - x)^2 + (x - b)^2 = (a - b)^2$.

26. $a^2 + b^2 + x^2 = 2(ax + bx + ab).$

27. $\dfrac{(x-a)^2 + (x-b)^2}{(x-a)^2 - (x-b)^2} = \dfrac{a^2+b^2}{a^2-b^2}.$

28. $x + \dfrac{1}{x} = \dfrac{5}{2}.$ 29. $x + \dfrac{2}{x} = 3.$

30. $x - \dfrac{1}{x} = \dfrac{15}{4}.$ 31. $x - \dfrac{2}{x} = 1.$

32. $x + \dfrac{1}{x} = \dfrac{m}{n}.$ 33. $x - \dfrac{a^2}{x} = c.$

In the following equations find the value of y in terms of x, and of x in terms of y:

34. $x^2 + xy + y^2 - c^2 = 0.$

Ans. $x = \dfrac{-y \pm \sqrt{4c^2 - 3y^2}}{2}$; $y = \dfrac{-x \pm \sqrt{4c^2 - 3x^2}}{2}.$

35. $x^2 - xy + y^2 - b^2 = 0.$ 36. $y^2 - 3xy + 2x^2 - x - y = c.$

37. $x^2 - 4xy + 4y^2 = 0.$ 38. $x^2 + nxy + ny^2 = 0.$

39. $a^2x^2 + abxy + b^2y^2 = 0.$ 40. $x^2 - 6\dfrac{m}{n}xy + 9\dfrac{m^2}{n^2}y^2 = 0.$

256. *Problems leading to Quadratic Equations.*

1. Find two numbers of which the sum is 32 and the product 231.

2. Find two numbers of which the sum is p and the product q.

3. Find two numbers of which the sum is 40 and the product 48 times the difference.

4. Of what two numbers is the difference 10 and the product 375?

5. Find those two numbers whose sum is 38 and the sum of whose squares is 724

6. Find those two numbers whose sum is m and the sum of whose squares is n^2.

7. Find those two numbers whose difference is m and the difference of whose squares is n^2.

8. Divide the number 26 into two parts whose product added to the sum of their squares shall be 556.

9. Divide 20 into two such parts that the square of their difference shall be equal to their product.

10. Divide 12 into two such parts that the sum of their squares shall be 4 times their product.

11. Can there be two unequal numbers the square of whose difference is equal to the difference of their squares?

12. A drover bought a certain number of sheep for \$663. Had he bought them for \$1 apiece less he would have got 48 more sheep. How many did he buy?

NOTE. If he bought x sheep, each sheep must have cost him $\frac{663}{x}$ dollars.

13. When tea costs 50 cents a pound more than coffee you can buy 20 pounds more of coffee than of tea for \$7.50. What is the price of each?

NOTE. After writing the equation simplify it by dividing out all common factors from its terms.

14. An almoner divided \$10 equally between two families. There was one member more in the second family than in the first, in consequence of which each member of the second got 25 cents less than each member of the first. What was the number of each family?

15. Find two numbers in the ratio 2 : 3 such that three times the lesser added to the square of the greater shall make 99. (§ 188, *Cor.*)

16. The length of a rectangular lot exceeds its breadth by 55 feet, and it contains 1564 square feet. What are its length and breadth?

17. A rectangular field takes 120 yards of fence to surround it, and contains 3375 square yards. What are its length and breadth?

18. A lot 120 feet by 25 feet is to be surrounded by a border of uniform breadth containing 250 square feet. What is the breadth of the border? (Express the result in decimals of an inch.)

19. The outer walls of a house 20 feet × 40 feet on the outside cover 224 square feet. What is their thickness?

20. It is shown in geometry that the area of a triangle is equal to half the product of its base by its altitude. From this theorem find the base of that triangle whose altitude exceeds its base by 3 feet, and whose area is 35 square feet.

SECTION III. EQUATIONS WHICH MAY BE SOLVED LIKE QUADRATICS.

257. Whenever an equation contains only two powers of the unknown quantity, and the index of one power is double that of the other, the equation can be solved as a quadratic.

Special Example. Let us take the equation

$$x^6 + bx^3 + c = 0. \tag{1}$$

Transposing c and adding $\frac{1}{4}b^2$ to each side of the equation, it becomes

$$x^6 + bx^3 + \frac{1}{4}b^2 = \frac{1}{4}b^2 - c.$$

The first member of this equation is a perfect square, namely, the square of $x^3 + \frac{1}{2}b$. Extracting the square roots of both members, we have

$$x^3 + \frac{1}{2}b = \sqrt{\left(\frac{1}{4}b^2 - c\right)} = \pm \frac{1}{2}\sqrt{(b^2 - 4c)}.$$

Hence $$x^3 = \frac{1}{2}[-b \pm \sqrt{(b^2 - 4c)}].$$

Extracting the cube root, we have

$$x = \frac{1}{2^{\frac{1}{3}}}[-b \pm \sqrt{(b^2 - 4c)}]^{\frac{1}{3}}.$$

General Form. We now generalize this solution in the following way: Suppose we can reduce an equation to the form

$$ax^{2n} + bx^n + c = 0,$$

in which the exponent n may be any quantity whatever, entire or fractional.

Mult. by a, $$a^2x^{2n} + abx^n = -ac.$$

Compl. square, $$a^2x^{2n} + abx^n + \frac{1}{4}b^2 = \frac{1}{4}b^2 - ac.$$

Ext. root, $$ax^n + \frac{1}{2}b = \sqrt{\left(\frac{1}{4}b^2 - ac\right)} = \frac{\sqrt{(b^2 - 4ac)}}{2}.$$

$\therefore$ $$x^n = \frac{1}{2a}[-b \pm \sqrt{(b^2 - 4ac)}].$$

The equation is now reduced to the form of a binomial. Extracting the nth root, we have

$$x = \frac{1}{2^{\frac{1}{n}} a^{\frac{1}{n}}} [-b \pm \sqrt{(b^2 - 4ac)}]^{\frac{1}{n}}$$

$$= \left(\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\right)^{\frac{1}{n}}.$$

If the exponent n is a fraction, the same course may be followed.

Suppose, for example,

$$ax^{\frac{4}{3}} + bx^{\frac{2}{3}} + c = 0.$$

Multiplying by a and transposing, we have

$$a^2x^{\frac{4}{3}} + abx^{\frac{2}{3}} = -ac.$$

Adding $\frac{b^2}{4}$ to both members,

$$a^2x^{\frac{4}{3}} + abx^{\frac{2}{3}} + \frac{b^2}{4} = \frac{b^2}{4} - ac.$$

The left-hand member of this equation is the square of

$$ax^{\frac{2}{3}} + \frac{b}{2}.$$

Extracting the square root of both members,

$$ax^{\frac{2}{3}} + \frac{b}{2} = \left(\frac{b^2}{4} - ac\right)^{\frac{1}{2}} = \frac{(b^2 - 4ac)^{\frac{1}{2}}}{2};$$

whence $$x^{\frac{2}{3}} = \frac{-b \pm (b^2 - 4ac)^{\frac{1}{2}}}{2a}.$$

Raising both sides of this equation to the third power and extracting the square root, we have

$$x = \left[\frac{-b \pm (b^2 - 4ac)^{\frac{1}{2}}}{2a}\right]^{\frac{3}{2}}.$$

EXERCISES.

Solve the equations:

1. $x + 2x^{\frac{1}{2}} = 3.$
2. $x - 2x^{\frac{1}{2}} + 1 = 0.$
3. $x^4 + px^2 + q = 0.$
4. $x^6 - 2nx^3 + 3n^2 = 0.$
5. $x^3 - 4ax^{\frac{3}{2}} = 4c^2.$
6. $x^{\frac{2}{3}} - 2x^{\frac{1}{3}} = 7.$
7. $4x + 8x^{\frac{1}{2}} = 28.$
8. $x^{\frac{1}{2}} - 4x^{\frac{1}{4}} = 12.$
9. $4x^{\frac{1}{3}} - 10x^{\frac{1}{6}} = 6.$
10. $x^{\frac{1}{n}} - x^{\frac{1}{2n}} = 1.$
11. $\sqrt[4]{x} - 2 + \sqrt{x} = 0.$
12. $\sqrt[6]{x} + \sqrt[3]{x} = 1.$

258. The principle of the preceding method may be applied to expressions containing the unknown quantity when the exponent of one expression is double that of the other.

EXAMPLE. To solve

$$(x-a)^2 - 2b(x-a) = 7b^2$$

We might solve this equation in the usual way, but an easier way will be to first treat $x - a$ itself as the unknown quantity and solve accordingly. The square will be completed by adding b^2 to each member, giving

$$(x-a)^2 - 2b(x-a) + b^2 = 8b^2.$$

Ext. root, $\quad x - a - b = \sqrt{8}b = 2\sqrt{2}b.$

$\therefore \quad x = a + (1 + 2\sqrt{2})b.$

EXERCISES.

Solve the following equations:*

1. $(x-c)^2 + 2m(x-c) = 3m^2.$
2. $\left(\frac{x}{b} - 1\right)^2 - 2\left(\frac{x}{b} - 1\right) = 4.$
3. $\left(\frac{x+1}{c}\right)^2 + 4 \cdot \frac{x+1}{c} = 4.$
4. $(x^2 - c)^2 - 4a(x^2 - c) = 12a^2.$
5. $(x^2 + a^2)^2 + a^2(x^2 + a^2) = 6a^2.$
6. $4\frac{x^4}{m^2} - 8\frac{x^2}{m} = 1.$
7. $\left(\frac{x}{c} - m\right)^2 - 2\left(\frac{x}{c} - m\right) = 4.$
8. $(x-c)^4 - 2(x-c)^2 = 7.$
9. $(x-b)^6 - 4b^3(x-b)^3 = 8b^6.$
10. $x^2 - 7x - 24 + (x^2 - 7x + 18)^{\frac{1}{2}} = 0.$

259. *Process of Substitution.* Equations of the preceding class may often be solved by substituting a single symbol for the expression containing the unknown quantity.

EXAMPLE. Solve

$$\left(\frac{x}{c} - 1\right)^2 - 8n\left(\frac{x}{c} - 1\right) = 11n^2.$$

* If the pupil finds the management of these equations difficult, he may solve them by substitution, as in the next article.

Let us put $$u \equiv \frac{x}{c} - 1. \qquad (a)$$

The equation then becomes

$$u^2 - 8nu = 11n^2.$$

Compl. square, $u^2 - 8nu + 16n^2 = 11n^2 + 16n^2 = 27n^2$,

$$u - 4n = \sqrt{27}n = \pm 3\sqrt{3}n,$$

$$u = (4 \pm 3\sqrt{3})n.$$

Having thus found the value of u, we are to substitute its value in the assumed equation (a) in order to find the value of x. Thus we have

$$\frac{x}{c} - 1 = (4 \pm 3\sqrt{3})n,$$

$$\frac{x}{c} = 1 + (4 \pm 3\sqrt{3})n,$$

$$x = c[1 + (4 \pm 3\sqrt{3})n].$$

EXERCISES.

Solve the equations of the preceding section by substitution, and also the following:

1. $\left(\frac{x}{x+1}\right)^2 - 2\frac{x}{x+1} = 3.$
2. $(x^2 - b^2)^2 + 4h(x^2 - b^2) = 5h^2.$

SECTION IV. FACTORING QUADRATIC EXPRESSIONS.

260. *To form a Quadratic Equation having Any Given Roots.* Let us, as an example, see how to form an equation whose roots shall be 3 and 5. Such an equation must be satisfied when we put

$$x = 3 \quad \text{or} \quad x = 5; \qquad (a)$$

that is, $$x - 3 = 0,$$
or $$x - 5 = 0.$$

Multiplying the first members, we have

$$(x - 3)(x - 5) = 0,$$

an equation which is evidently satisfied by either of the values of x in (a).

Developing the product, the equation becomes

$$x^2 - 8x + 15 = 0.$$

NOTE. Let the pupil now solve this equation and show that 3 and 5 are really its roots.

EXERCISES.

Form equations of which the roots shall be:

1. $+3$ and -5. Ans. $x^2 + 2x - 15 = 0$.
2. -3 and $+5$.
3. $+1$ and -1.
4. -3 and -8.
5. -1 and $+6$.
6. $+3$ and $+7$.
7. $1 + 2\sqrt{3}$ and $1 - 2\sqrt{3}$.
8. $3 + \sqrt{2}$ and $3 - \sqrt{2}$.
9. $1 + \sqrt{5}$ and $1 - \sqrt{5}$.
10. $5 + \sqrt{5}$ and $5 - \sqrt{5}$.
11. $\frac{1}{2}$ and $\frac{3}{2}$.
12. $\frac{2}{3}$ and $-\frac{5}{3}$.
13. $\frac{1}{2} + \sqrt{\frac{1}{2}}$ and $\frac{1}{2} - \sqrt{\frac{1}{2}}$.

261. Let us form the equation of which the roots shall be α and β. Proceeding as before,

$$(x - \alpha)(x - \beta) = 0.$$

Developing, $$x^2 - (\alpha + \beta)x + \alpha\beta = 0.$$

Here, the coefficient of x being $-(\alpha + \beta)$, we see that a quadratic equation with given roots is formed by the principle:

The coefficient of x^2 is unity.

That of x is the negative of the sum of the roots.

The term not containing x is the product of the roots.

The same is true of any quadratic equation.

For we have found the roots of the general equation

$$x^2 + px + q = 0$$

to be $$-\tfrac{1}{2}p - \tfrac{1}{2}\sqrt{(p^2 - 4q)}$$

and $$-\tfrac{1}{2}p + \tfrac{1}{2}\sqrt{(p^2 - 4q)}.$$

The sum of these quantities we find to be $-p$, and by multiplying we shall find their product to be q, thus proving the proposition.

262. *Factoring a Quadratic Expression.* As we have reduced the product

$$(x - \alpha)(x - \beta)$$

to the form $x^2 - (\alpha + \beta)x + \alpha\beta$,

so, conversely, we may divide the general expression

$$x^2 + px + q \qquad (b)$$

into two factors. In this expression we need not consider x to represent an unknown quantity, but to stand, like any other symbol, for any quantity we please.

We may then transform this expression as in the following examples:

1. Factor $x^2 + 2x - 3$.

We note that by § 241 the first two terms, increased by unity, will form a perfect square. So we add and subtract 1, and then factor by § 238, thus:

$$\begin{aligned} x^2 + 2x - 3 &= x^2 + 2x + 1 - 1 - 3 \\ &= (x + 1)^2 - 4 \\ &= (x + 1 + 2)(x + 1 - 2) \\ &= (x + 3)(x - 1). \end{aligned}$$

2. Factor $x^2 - 4x + 1$.

Adding and subtracting 4, we have

$$\begin{aligned} x^2 - 4x + 1 &= x^2 - 4x + 4 - 4 + 1 \\ &= (x - 2)^2 - 3 \\ &= (x - 2 + \sqrt{3})(x - 2 - \sqrt{3}). \end{aligned}$$

EXERCISES.

Factor:

1. $x^2 - 2x - 1$.
2. $x^2 + 2x - 1$.
3. $x^2 + 4x + 2$.
4. $x^2 + 6x - 7$.
5. $r^2 - 6r + 1$.
6. $r^2 - 10r + 5$.
7. $p^2 + 8p - 8$.
8. $r^2 - 8r - 2$.

263. Let us now apply the same process to the general form (b). We may transform it as follows:

$$\left.\begin{aligned} x^2 + px + q &= x^2 + px + \tfrac{1}{4}p^2 - \tfrac{1}{4}p^2 + q \\ &= (x + \tfrac{1}{2}p)^2 - (\tfrac{1}{4}p^2 - q) \\ &= (x + \tfrac{1}{2}p)^2 - \tfrac{1}{4}(p^2 - 4q) \end{aligned}\right\} \quad (\S\ 241)$$

$$= \{x + \tfrac{1}{2}p + \tfrac{1}{2}\sqrt{(p^2 - 4q)}\}\{x + \tfrac{1}{2}p - \tfrac{1}{2}\sqrt{(p^2 - 4q)}\}. \quad (\S\ 238)$$

If we now seek the roots of the equation formed by equating these expressions to zero, we have the general principle :

In order that a product may be equal to zero at least one of the factors must be zero.

Hence we must have either

$$x + \tfrac{1}{2}p + \tfrac{1}{2}\sqrt{(p^2 - 4q)} = 0,$$

whence $$x = -\tfrac{1}{2}p - \tfrac{1}{2}\sqrt{(p^2 - 4q)},$$

or $$x + \tfrac{1}{2}p - \tfrac{1}{2}\sqrt{(p^2 - 4q)} = 0,$$

whence $$x = -\tfrac{1}{2}p + \tfrac{1}{2}\sqrt{(p^2 - 4q)}.$$

Hence *we may find the roots of any quadratic equation in the general form by factoring the expression which the equation states to be zero.*

EXERCISES.

Find the roots of the following equations by factoring:

1. $x^2 - 3x = 0.$
2. $x^2 - ax = 0.$
3. $x^2 - (a + b)x = 0.$
4. $x^2 - 2x - 1 = 0.$
5. $(x - 1)^2 = a(x^2 - 1).$
6. $x^2 + 2x - 4 = 0.$
7. $x^2 + 6x + 8 = 0.$
8. $x^2 + x - 1 = 0.$
9. $x^2 - x - 1 = 0.$
10. $x^2 + \tfrac{1}{2}x - 2 = 0.$
11. $x^2 - \tfrac{1}{3}x - 1 = 0.$
12. $x^2 - ax - b = 0.$
13. $x^2 - ax + b = 0.$
14. $x^2 + ax - b = 0.$
15. $x^2 + ax + b = 0.$
16. $x^2 + xy - y^2 = 0.$
17. $x^2 + mxy + y^2 = 0.$
18. $x^2 - mxy + y^2 = 0.$

264. The principle of § 261 enables us to determine the signs of the two roots of a quadratic equation by inspection.

1. Because in the equation

$$x^2 + px + q = 0$$

the term q is the product of the roots,

q is positive when the roots have like signs, and negative when they have unlike signs.

2. Because p is the *negative* of the sum of the roots,

At least one of the roots must be negative when p is positive, and positive when p is negative.

Hence in such an equation as

$$x^2 + mx + n = 0$$

the roots, being like in signs and one at least negative, must both be negative.

In such an equation as

$$x^2 - mx + n = 0$$

both roots must, for a similar reason, be positive.

In such an equation as

$$x^2 \pm mx - n = 0$$

one root must be positive and the other negative.

265. *Other Relations between the Roots and the Coefficients of a Quadratic Equation.* The preceding theory will enable us to express many functions of the roots in terms of the coefficients without solving the equation.

EXAMPLE 1. Find the sum of the squares of the roots of the equation

$$x^2 + 5x + 2 = 0. \qquad (a)$$

Solution. If we put

$$\alpha \equiv \text{the one root,}$$
$$\beta \equiv \text{the other root,}$$

we have, by § 261,

$$\alpha + \beta = -5, \qquad (b)$$
$$\alpha\beta = +2. \qquad (c)$$

Squaring the first of these equations, doubling the second and subtracting, we have

$$\begin{array}{rcl} \alpha^2 + 2\alpha\beta + \beta^2 &=& 25 \\ 2\alpha\beta \qquad &=& 4 \\ \hline \alpha^2 \qquad\quad + \beta^2 &=& 21. \quad \text{Ans.} \end{array}$$

NOTE. The student may verify this result by solving the equation.

Ex. 2. Find the sum of the cubes of the roots of the above equation.

Solution. Cubing the equation (*b*), we have

$$\alpha^3 + 3\alpha^2\beta + 3\alpha\beta^2 + \beta^3 = -125.$$

The two middle terms of the first member may be put into the form

$$3\alpha^2\beta + 3\alpha\beta^2 = 3\alpha\beta(\alpha + \beta).$$

Substituting the values of the factors from (*b*) and (*c*), we have

$$3\alpha\beta(\alpha + \beta) = -30.$$

$$\therefore \quad \alpha^3 - 30 + \beta^3 = -125,$$
$$\alpha^3 + \beta^3 = -95. \quad \text{Ans.}$$

EXERCISES.

Find the values of the following functions of the roots of equation (a):

1. $\alpha^2 + \alpha\beta + \beta^2$. 2. $\alpha^2 - \alpha\beta + \beta^2$.
3. $\alpha + \alpha^2\beta^2 + \beta$. 4. $\alpha^3 - 3\alpha\beta + \beta^3$.

Prove that if x_1 and x_2 are the roots of the equation

$$x^2 + px + q = 0,$$

we shall have:

5. $x_1^2 + x_2^2 = p^2 - 2q$.
6. $x_1^2 + x_1x_2 + x_2^2 = p^2 - q$.
7. $x_1^2 - x_1x_2 + x_2^2 = p^2 - 3q$.
8. $x_1^3 + x_2^3 = 3pq - p^3$.

Find the values of:

9. $x_1^3 + x_1^2x_2 + x_1x_2^2 + x_2^3$.
10. $x_1^3 - x_1^2x_2 - x_1x_2^2 + x_2^3$.

SECTION V. SOLUTION OF IRRATIONAL EQUATIONS.

266. An **irrational equation** is one in which the unknown quantity appears under the radical sign.

An irrational equation may be cleared of fractions in the same way as if it were rational.

EXAMPLE. Clear from fractions the equation

$$\frac{a}{\sqrt{x+c}} - \frac{b}{\sqrt{x-c}} = \frac{ab}{\sqrt{x^2-c^2}}. \qquad (a)$$

The L.C.M. of the denominators is $\sqrt{(x+c)(x-c)} = \sqrt{x^2 - c^2}$. Multiplying all the terms by this factor, the equation becomes

$$a\sqrt{x-c} - b\sqrt{x+c} = ab, \qquad (b)$$

and the equation is cleared of fractions.

267. To reduce an irrational equation to a rational one.

RULE. 1. *Clear the equation of fractions.*

2. *Transpose terms so that a surd expression shall be a factor of one member of the equation.*

3. *Square both members so as to rationalize the surd member of the equation.*

4. *Repeat* (2) *and* (3) *until no surds containing the unknown quantity are left.*

EXAMPLE. To rationalize the equation (b).

Transposing, $a\sqrt{x-c} = b\sqrt{x+c} + ab.$

Squaring, $a^2(x-c) = b^2(x+c) + a^2b^2 + 2ab^2\sqrt{x+c}.$

Transposing, $a^2(x-c) - b^2(x+c) + a^2b^2 = 2ab^2\sqrt{x+c},$

or $(a^2 - b^2)x - a^2c - b^2c + a^2b^2 = 2ab^2\sqrt{x+c}.$

Squaring, $(a^2 - b^2)^2x^2 + 2(a^2b^2 - a^2c - b^2c)(a^2 - b^2)x$
$+ (a^2b^2 - a^2c - b^2c)^2 = 4a^2b^4(x+c),$

a rational equation.

EXERCISES.

Reduce and solve the following equations:

1. $\dfrac{3}{\sqrt{x+4}} + \dfrac{1}{\sqrt{x-4}} = \dfrac{6}{\sqrt{x^2-16}}.$

Principal Steps.

$$3\sqrt{x-4} + \sqrt{x+4} = 6.$$
$$9(x-4) = 40 + x - 12\sqrt{x+4}.$$
$$9x + 36 = 361 - 76x + 4x^2.$$
$$x = 5 \quad \text{or} \quad \frac{65}{4}.$$

2. $(x+4)^{\frac{1}{2}} + (x-3)^{\frac{1}{2}} = 7.$

3. $(x+2)^{\frac{1}{2}} + 2(x+7)^{\frac{1}{2}} = 8.$

4. $\dfrac{4c}{\sqrt{x^2+c}} = x + \sqrt{x^2+c}.$

5. $\sqrt{x+a} - \sqrt{x-a} = \sqrt{x}.$

6. $\dfrac{a+\sqrt{x^2-ax}}{a-\sqrt{x^2-ax}} = 3.$

7. $\dfrac{1-\sqrt{x+1}}{1+\sqrt{x-1}} + 1 = 0.$

8. $\dfrac{\sqrt{x^2+a^2}+a}{\sqrt{x^2+a^2}-a} = n.$

9. $\sqrt{x^2+a^2}+\sqrt{x^2-a^2}=2a.$

10. $\sqrt{a^2+x^2}+\sqrt{a^2-x^2}=a.$

11. $\dfrac{1}{\sqrt{x^2-a^2}-a}+\dfrac{1}{\sqrt{x^2-a^2}+a}=\dfrac{1}{a}.$

12. $\sqrt{x-8}-\sqrt{x-3}=\sqrt{x}.$

13. $\sqrt{(x+4)}-\sqrt{\left(x+\frac{3}{2}\right)}=\sqrt{x}.$

14. $\sqrt{(4x+21)}+\sqrt{(x+3)}=\sqrt{(x+8)}.$

15. $2\sqrt{x}=1+\sqrt{4x+\sqrt{(7x+2)}}.$

16. $\sqrt{(a+x)}+\sqrt{(a-x)}=2\sqrt{a}.$

17. $\dfrac{\sqrt{(a+x)}}{\sqrt{a}+\sqrt{(a+x)}}=\dfrac{\sqrt{(a-x)}}{\sqrt{a}-\sqrt{(a-x)}}.$

18. $\sqrt{x}\{\sqrt{(a-x)}-\sqrt{(a+x)}\}=\sqrt{a}\{\sqrt{(a^2-x^2)}-a\}.$

19. $\sqrt{a^2-x}+\sqrt{b^2+x}=a+b.$

20. $\sqrt{a-x}+\sqrt{b-x}=\sqrt{a+b}.$

21. $\sqrt{a-x}+\sqrt{x-b}=\sqrt{a+b-2x}.$

22. $\dfrac{\sqrt{a-x}+\sqrt{x-b}}{\sqrt{a-x}-\sqrt{x-b}}=\sqrt{\dfrac{a-x}{x-b}}.$

23. $\dfrac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}}=\dfrac{a}{b}.$

Express the first member as a ratio and reduce by composition and division.

24. $\sqrt{4a+b-4x}-2\sqrt{a+b-2x}=\sqrt{b}.$

25. $a\sqrt{m+x}-b\sqrt{m-x}=\sqrt{m(a^2+b^2)}.$

26. $\sqrt{(a+x)(x+b)}+\sqrt{(a-x)(x-b)}=2\sqrt{ax}.$

SECTION VI. SIMULTANEOUS QUADRATIC AND OTHER EQUATIONS.

268. *Def.* Two equations, one or both of the second degree, between two unknown quantities, are called a pair of **simultaneous quadratic equations.**

The most general form of an equation of the second degree between the unknown quantities x and y is

$$ax^2 + bxy + cy^2 + dx + ey + f = 0.$$

If we eliminate one unknown quantity between two such equations, the resulting equation, when reduced, will be of the fourth degree with respect to the other unknown quantity, and therefore cannot in general be solved as a quadratic. But there are several cases in which a solution of two equations, one of which is of the second or some higher degree, may be effected, owing to some special relations among the coefficients of the unknown quantities in one or both equations. The following are the most common:

269. CASE I. *When one of the equations is of the first degree only.*

This case may be solved thus:

RULE. *Find the value of one of the unknown quantities in terms of the other from the equation of the first degree. This value being substituted in the other equation, we shall have a quadratic equation from which the other unknown quantity may be found.*

EXAMPLE. Solve

$$\left.\begin{aligned} x^2 - 2y^2 &= a, \\ 2x - 3y &= b. \end{aligned}\right\} \qquad (a)$$

From the second equation we find

$$x = \frac{3y + b}{2}; \qquad (b)$$

whence $$x^2 = \frac{9y^2 + 6by + b^2}{4}.$$

Substituting this value in the first equation (a) and reducing, we find

$$y^2 + 6by + 9b^2 = 4(a + 2b^2).$$

Solving this quadratic equation,

$$y = -3b \pm 2\sqrt{a + 2b^2}.$$

This value of y being substituted in the equation (b) gives

$$x = -4b \pm 3\sqrt{a + 2b^2}.$$

The same problem may be solved in the reverse order by eliminating y instead of x. The second equation (a) gives

$$y = \frac{2x - b}{3}.$$

If we substitute this value of y in the first equation, we shall have a quadratic equation in x, from which the value of the latter quantity can be found.

The result should be proved by substituting the values of x and y in the first equation.

EXERCISES.

Solve:

1. $x + y = a$; $\quad xy = b^2$.
2. $x - y = a$; $\quad \frac{x}{y^2} = n$.
3. $x + y = 3$; $\quad x^2 + y^2 = 29$.
4. $x - y = 3$; $\quad x^2 - y^2 = 51$.
5. $x + y = c$; $\quad x^2 - y^2 = g^2$.
6. $x - y = 2c$; $\quad x^2 + y^2 = 2b^2$.
7. $x - 3y = 1$; $\quad x^2 - 3y^2 = 121$.
8. $x + \frac{1}{y} = 7$; $\quad x^2 - \frac{1}{y^2} = 21$.

   Consider $\frac{1}{y}$ as an unknown quantity.
9. $x - \frac{1}{y} = 10$; $\quad x^2 + \frac{1}{y^2} = 58$.
10. $y - \frac{1}{x} = 11$; $\quad y^2 + \frac{1}{x} = 145$.
11. $y^2 + \frac{1}{x^2} = 29$; $\quad y - \frac{1}{x} = 3$.
12. $xy = c^2$; $\quad \frac{x}{y} = m$.

13. $\dfrac{x+y}{x-y}=\dfrac{a}{b}$; $\quad x^2+y^2=c^2$.

14. $\dfrac{x+y}{x-y}=\dfrac{a}{b}$; $\quad ax^2+(a-b)xy-by^2=c^2$.

15. $\dfrac{ax-by}{cx-dy}=1$; $\quad x^2+y^2=k^2$.

16. $\dfrac{(a-x)^2+y^2}{(a-x)y}=\dfrac{13}{6}$; $\quad x-y=m$.

17. $x+y=a$; $\quad \dfrac{x}{b-y}+\dfrac{b-y}{x}=\dfrac{5}{2}$.

18. $x+y=c$; $\quad x^2+y^2=mxy$.

270. CASE II. If each equation contains only one term of the second degree, and these terms are similar, they may be eliminated by the method of addition and subtraction, leading to an equation of the first degree. We may then proceed as in Case I.

EXAMPLE. Solve

$$\begin{aligned} 2xy - x - y &= 4, \\ -5xy + 2x + 3y &= -6. \end{aligned}$$

Multiplying the first equation by 5 and the second by 2, and adding, we have

$$\begin{array}{r} 10xy - 5x - 5y = 20 \\ -10xy + 4x + 6y = -12 \\ \hline -x + y = 8 \end{array}$$

Hence $\qquad y = 8 + x. \qquad (a)$

Substituting this value of y in the first of the given equations and reducing, we find

$$2x^2 + 14x - 8 = 4,$$

or $\qquad x^2 + 7x = 6.$

Solving this quadratic, we find

$$x = -\tfrac{7}{2} \pm \tfrac{1}{2}\sqrt{73};$$

whence, from (a), $\qquad y = \tfrac{9}{2} \pm \tfrac{1}{2}\sqrt{73}.$

EXERCISES.

Solve:

1. $x^2 - y = 1; \qquad x^2 + 3y - x = 2.$
2. $x + y + x^2 = 22; \qquad x - 5y + x^2 = 10.$
3. $x - y + xy = a; \qquad x + y - xy = 3a.$
4. $x^2 + 3xy + 3y = 43; \qquad x^2 + 3xy - 3x = 25.$
5. $x = a(x^2 + y^2); \qquad y = b(x^2 + y^2).$
6. $x = a\sqrt{x + y}; \qquad y = b\sqrt{x + y}.$

271. CASE III. When the functions of the unknown quantities in the two equations contain one or more common factors, we may sometimes form an equation of lower degree by taking the quotient of the two equations and dividing out the common factors.

EXAMPLE. $\quad x^2 + xy = a; \qquad y^2 + xy = b.$

Factoring, we have

$$x(x + y) = a,$$
$$y(x + y) = b;$$

whence, by taking the quotient,

$$\frac{x}{y} = \frac{a}{b}, \quad \text{or} \quad y = \frac{b}{a}x.$$

The last equation is of the first degree, and by substitution we readily find

$$x = \pm \frac{a}{\sqrt{a + b}}; \qquad y = \pm \frac{b}{\sqrt{a + b}}.$$

EXERCISES.

Solve:

1. $x^3 + xy^2 = a; \qquad y^3 + x^2y = b.$
2. $x^2y + xy^2 = a; \qquad x^2y - xy^2 = b.$
3. $x\sqrt{x + y} = a^{\frac{3}{2}}; \qquad y\sqrt{x + y} = b^3.$
4. $x + y = m(x^2 + y^2); \qquad x - y = n(x^2 + y^2).$
5. $x + y = m(x^2 - y^2); \qquad x - y = n(x^2 - y^2).$
6. $x^4 + xy^3 = a; \qquad y^4 + x^3y = b.$
7. $x + y = 7; \qquad x^3 + y^3 = 133.$
8. $xy + y^2 = a; \qquad x^2 - y^2 = b.$

272. Case IV. The solution may often be simplified by combining the two equations so as to get others simpler in form or admitting of being factored.

The following is an elegant example:

$$x^2 + y^2 = a,$$
$$xy = b.$$

Take twice the second equation, and add and subtract it from the first. Then

$$x^2 + 2xy + y^2 = a + 2b, \quad \text{or} \quad (x + y)^2 = a + 2b;$$
$$x^2 - 2xy + y^2 = a - 2b, \quad \text{or} \quad (x - y)^2 = a - 2b.$$

Ext. root,
$$x + y = \sqrt{a + 2b}$$
$$x - y = \sqrt{a - 2b}$$
$$2x = \sqrt{a + 2b} + \sqrt{a - 2b}$$
$$2y = \sqrt{a + 2b} - \sqrt{a - 2b}$$

Dividing by 2, we have the values of x and y.

As another example, take the equations

$$x^2 + y = nx,$$
$$y^2 + x = ny.$$

If we take their difference we shall find it divisible by $x - y$, giving an equation which leads to

$$x + y = n + 1,$$

or
$$y = n + 1 - x.$$

Substituting this value of y in the first equation, we have a quadratic in x.

EXERCISES.

Solve:

1. $x^2 + 4y^2 = 4c^2; \qquad xy = h.$
2. $x^2 - xy = a^2; \qquad y^2 - xy = b^2.$
3. $x^2 + y^2 + x + y = 62; \qquad x^2 + y^2 - x - y = 44.$
4. $x^2 + y^2 - x + y = 49; \qquad x^2 + y^2 + x - y = 33.$
5. $x^2 - 3xy = 72; \qquad xy - 4y^2 = 8.$
6. $x^2 - y^2 + x - y = 2m; \qquad x^2 - y^2 - x + y = 2n.$

The following equations may be solved by various combinations of the preceding methods:

7. $\dfrac{x}{y} + \dfrac{y}{x} = \dfrac{10}{3}; \qquad x^2 + y^2 = 40.$

8. $x^2 + y^2 = 74;\qquad x + y + xy = 47.$

To solve this form a quadratic with $x + y$ as the unknown quantity.

9. $x^2 + y^2 = 85;\qquad x - y + xy = 25.$

10. $x^2 + y^2 + x + y = 48;\qquad 2(x + y) = 3xy.$

11. $x^2 - xy + y^2 = 2a^2;\qquad x^2 + xy + y^2 = 2c^2.$

12. $x + y^2 = ax;\qquad x^2 + y = by.$

13. $x^2 + y\sqrt{xy} = 36;\qquad y^2 + x\sqrt{xy} = 72.$

☞ Teachers requiring problems leading to equations with two or more unknown quantities will find them in the Appendix.

Section VII. Imaginary Roots.

273. Since the squares of both negative and positive quantities are positive, the square root of a negative quantity cannot be any quantity which we have hitherto considered.

Def. The indicated square root of a negative quantity is called an **imaginary quantity**.

The term *imaginary* is applied because in this case we have to *imagine* or *suppose* a quantity of which the square shall be negative.

Def. The positive and negative quantities of algebra are called **real**.

274. Theorem. *An imaginary quantity can always be expressed as the product of a real quantity into the square root of* -1.

Proof. Let $-a$ be the negative quantity whose root is to be expressed. Because

$$-a = a \times -1,$$

we have, by § 227,

$$\sqrt{-a} = \sqrt{a} \times \sqrt{-1}.$$

Because a is positive, $\sqrt{a}$ is real, so that the theorem is proved.

Notation. It is common to use the symbol i for $\sqrt{-1}$, so that

$$i \equiv \sqrt{-1}.$$

This is because it is easier to write i than $\sqrt{-1}$.

EXERCISES.

Express the square roots:

1. $\sqrt{-4}$. Ans. $2\sqrt{-1}$ or $2i$.

2. $\sqrt{-25}$. 3. $\sqrt{-\frac{16}{49}}$.

4. $\sqrt{-k^2}$. 5. $\sqrt{-k^4}$.

6. $\sqrt{-4k}$. Ans. $2\sqrt{k}\sqrt{-1}$ or $2k^{\frac{1}{2}}i$.

7. $\sqrt{-9h}$. 8. $\sqrt{-36h^2}$.

9. $\sqrt{-20}$. Ans. $2\,.\,5^{\frac{1}{2}}\sqrt{-1}$ or $2\,.\,5^{\frac{1}{2}}i$.

10. $\sqrt{-12k^2}$.

Extract the square roots of:

11. $-a^2 + 2ab - b^2$. Ans. $(a - b)i$.

12. $-a^2 + 4ab - 4b^2$. 13. $-m^2 - 2mn - n^2$.

14. $-\frac{m^2}{n^2} + 2\frac{m}{n} - 1$. 15. $-\frac{m^2}{n^2} - 2\frac{m}{n} - 1$.

275. Since the roots of the general quadratic equation

$$x^2 + px + q = 0$$

contain the expression $\sqrt{p^2 - 4q}$, it follows that if the quantity under the radical sign is negative there is no real root. In such cases the algebraic solution may be expressed by imaginary quantities.

EXERCISES.

Solve the following quadratic equations:

1. $x^2 - 2x = -5$. Ans. $x = 1 \pm 2\sqrt{-1} = 1 \pm 2i$.

2. $x^2 - 4x = -5$.

3. $x^2 - 2ax = -2a^2$.

4. $x^2 + 4ax + 8a^2 = 0$.

276. When the solution of a problem contains an imaginary quantity it shows that the conditions of the problem are impossible.

Example. To find two numbers of which the sum shall be 6 and the product 10.

Solving in the usual way, we may put

$$x \equiv \text{one number};$$
$$6 - x \equiv \text{the other}.$$

Then
$$x(6 - x) = 10,$$
$$x^2 - 6x = -10,$$
$$x^2 - 6x + 9 = -1,$$
$$x = 3 \pm \sqrt{-1} = 3 \pm i.$$

The answer being imaginary, there are no numbers which fulfil the conditions. Indeed, by assigning to x different values from 0 to 6 we see that 9 is the greatest possible product of two numbers whose sum is 6. If one number is less than 0 or more than 6 the product will be negative.

EXERCISES.

1. Find two numbers of which the sum shall be 8 and the sum of the squares n. Then show that if n is less than 32 the condition is impossible in real numbers.

Method of Solution. If x and $8 - x$ represent the numbers, the conditions of the problem give
$$x^2 + (8 - x)^2 = n,$$
or
$$2x^2 - 16x + 64 = n.$$
Solving this quadratic equation, we find the roots to be
$$x = 4 \pm \sqrt{\frac{n - 32}{2}}.$$

If $n < 32$, the quantity under the radical sign is negative, showing that there is no real solution.

2. Find two numbers of which the difference shall be 4 and the sum of the squares m, and show what is the least possible value of m for which the problem is possible.

3. If the sum of two numbers is s and the sum of their squares q, what is the least possible value of q?

4. If the sum of two numbers is 10 and their product p, what is the greatest possible value of p?

5. If the sum of two numbers is s, what is the greatest possible value of their product?

6. If the difference of two numbers is d, what is the least possible value of the sum of their squares?

277. REMARK. The so-called imaginary quantities are *unreal* only in the sense that they cannot be represented by the ordinary positive and negative numbers of algebra. They are, from a purely algebraic point of view, as real as other algebraic quantities, and are of the greatest use in the higher mathematics.

MEMORANDA FOR REVIEW.

Define: Quadratic equation; Pure quadratic; Complete quadratic; Binomial equation; Normal form; General form; Coefficients.

Pure Equations.

Method of solving a binomial equation when the unknown quantity appears under the sign of $\left\{\begin{array}{l}\text{a power;}\\ \text{a root;}\\ \text{a power of a root.}\end{array}\right.$

Complete Quadratic Equations.

Explain

That the equation, if in the normal form, may be reduced by multiplication so that the term containing the square of the unknown quantity as a factor shall itself be a perfect square.

That the quadratic expression may then, by aggregation in parentheses, be reduced to the form of a binomial, one of whose terms is a perfect square, while the other term does not contain the unknown quantity.

That we may then proceed by two methods:

I. Transpose the second term and extract root.

II. Factor the binomial and place each factor equal to zero.

That by $\left\{\begin{array}{l}\text{I.}\\ \text{II.}\end{array}\right.$ we obtain $\left\{\begin{array}{l}\text{one equation}\\ \text{two equations}\end{array}\right.$ of the first degree in the unknown quantities, and thus, by either method, obtain the same pair of values of the unknown quantities.

To form a quadratic equation with given roots.
Expressions of the coefficients in terms of the roots.
Rule for signs of roots.

Equations which may be solved as Quadratics.

Treatment of the equation when it contains only one function of the unknown quantity, and the exponent of this function in one term is double its exponent in the other term.

Illustrate by examples, and give solution:

I. Leaving the function unchanged.
II. Substituting a symbol for it.

Irrational Equations.

Give rule for solving.

Simultaneous Quadratic Equations.

Case I. Form; Rule.
Case II. Form; Rule.
Case III. Form; Rule.
Case IV. Form; Rule.

Imaginary Roots.

Define Imaginary unit.

Reduction of the square root of a negative quantity to a number of imaginary units; Method.

THIRD COURSE.

PROGRESSIONS VARIATIONS AND LOGARITHMS.

CHAPTER I.

PROGRESSIONS.

SECTION I. ARITHMETICAL PROGRESSION.

278. *Def.* An **arithmetical progression** is a series of terms each of which is greater or less than the preceding by a constant quantity.

EXAMPLE. The series

$$3, \quad 5, \quad 7, \quad 9, \quad 11, \quad 13, \quad \text{etc.};$$
$$7, \quad 4, \quad 1, \quad -2, \quad -5, \quad \text{etc.};$$
$$a+b, \quad a, \quad a-b, \quad a-2b, \quad a-3b, \quad \text{etc.},$$

are each in arithmetical progression, because, in the first, each term is greater than the preceding by 2; in the second, each is less than the preceding by 3; in the third, each is less than the preceding by b.

Def. The amount by which each term of an arithmetical progression is algebraically greater than the preceding one is called the **common difference.**

Def. All the terms of an arithmetical progression except the first and last are called **arithmetical means** between the first and last as extremes.

Cor. The arithmetical mean of two quantities is half their sum.

EXAMPLE. The four numbers 5, 8, 11, 14 form four arithmetical means between 2 and 17.

NOTE. We put for brevity A.P. $\equiv$ arithmetical progression.

EXERCISES.

1. Write the first seven terms of the progression of which the first term is 8 and the common difference -3.

2. Write five terms of the progression of which the first term is $a - 6n$ and the common difference $2n$.

3. Write a progression of five terms of which the middle term is x and the common difference d.

4. Prove that if three terms are in arithmetical progression the middle one is half the sum of the extremes.

5. Prove that if four terms are in A.P. the sum of the extremes is equal to the sum of the means.

6. Prove that the sum of five terms in A.P. is five times the middle term.

In the last three proofs choose some symbol for the common difference, and any expression you find most convenient for the first term.

Problems in Arithmetical Progression.

Let us put

a, the first term of a progression;
d, the common difference;
n, the number of terms;
l, the last term;
Σ, the sum of all the terms.

The series is then

$$a, \quad a+d, \quad a+2d, \; . \; . \; . \; . \; l.$$

Any three of the above five quantities being given, the other two may be found.

279. Problem I. *Given the first term, the common difference and the number of terms, to find the last term.*

The first term is here a,
second " " " $a+d$,
third " " " $a+2d$.

The coefficient of d is, in each case, 1 less than the number of the term. Since this coefficient increases by unity for every term we add, it must remain less by unity than the number of the term. Hence, whatever be the value of i,

The ith term is $a+(i-1)d$.

Hence when $i=n$,

$$l = a + (n-1)d. \qquad (1)$$

From this equation we can solve the further problems:

280. Problem II. *Given the last term l, the common difference d and the number of terms n, to find the first term.*

The solution is found by solving (1) with respect to a, which gives

$$a = l - (n - 1)d. \qquad (2)$$

281. PROBLEM III. *Given the first and last terms, a and l, and the number of terms n, to find the common difference.*

Solution from (1), d being the unknown quantity,

$$d = \frac{l - a}{n - 1}. \qquad (3)$$

282. PROBLEM IV. *Given the first and last terms and the common difference, to find the number of terms.*

Solution, also from (1),

$$n = \frac{l - a}{d} + 1 = \frac{l - a + d}{d}. \qquad (4)$$

283. *Generalization of the Preceding Problems.* The solutions of these four problems all depend upon the one general principle:

The difference between any two terms of an arithmetical progression is equal to the common difference multiplied by the difference of their numbers in order.

By the number in order of a term we mean the numeral which expresses its place in the series, as the first, second, third, etc.

EXAMPLE. The difference between the second and the ninth terms is $(9 - 2)d = 7d$.

The difference between the first and the nth is $(n - 1)d$.

EXERCISES.

In arithmetical progressions there are:

1. Given common difference $+5$, third term $= 14$; find first term. *Ans.* First term $= 4$.

2. Given fourth term $= b$, common difference $= -3$; find first seven terms.

3. Given third term $= a + x$, fourth term $= a + 2x$; find first five terms.

4. Given first term $= a - b$, sixth term $= 6a + 4b$; write the six terms.

5. Given fifth term $= 7x - 5y$, seventh term $= 9x - 9y$; find first seven terms and common difference.

6. If the first term is 7 and the common difference 5, what is the twelfth term?

7. Given first term $2a$, tenth term $11a - 18b$; find common difference.

8. Given first term a, fifth term b; write the three intervening terms.

9. Given tenth term $= 12$, twelfth term $= 8$; find first term.

10. Given first term 1, last term 153, common difference 4; find the number of terms.

284. PROBLEM V. *To find the sum of all the terms of an arithmetical progression.*

We have, by the definition of Σ,

$$\Sigma = a + (a + d) + (a + 2d) + \ldots\ldots (l - d) + l,$$

the parentheses being used only to distinguish the terms.

Now let us write the terms in reverse order. The term before the last is $l - d$, the second one before it $l - 2d$, etc.

We therefore have

$$\Sigma = l + (l - d) + (l - 2d) \ldots\ldots + (a + d) + a.$$

Adding these two values of Σ together, term by term, we find

$$2\Sigma = (a + l) + (a + l) + (a + l) + \ldots\ldots + (a + l) + (a + l),$$

the quantity $(a + l)$ being written as often as there are terms; that is, n times. Hence

$$2\Sigma = n(a + l),$$

$$\Sigma = n\frac{a + l}{2}. \qquad (5)$$

REMARK. The expression $\frac{a + l}{2}$, that is, half the sum of the extreme terms, is the *mean value* of all the terms. The sum of the n terms is therefore the same as if each of them had this value.

285. In the equation (5) we are supposed to know the first and last terms and the number of terms. If other quantities are taken as the known ones, we have to substitute for some one of the quantities in (5) its value expressed by one of the equations (1), (2), (3) or (4). Suppose, for example, that

we have given only the last term, the common difference and the number of terms; that is, l, d and n. We must then in (5) substitute for a its value in (2). This will give

$$\Sigma = n\left(l - \frac{n-1}{2}d\right) = nl - \frac{n(n-1)}{2}d. \qquad (6)$$

EXERCISES.

1. Find the sum of the first 100 numbers:

$$1 + 2 + 3 + 4 + \ldots . + 100.$$

2. Find the sum of the odd numbers to 99:

$$1 + 3 + 5 + 7 + \ldots . + 99.$$

3. Find the sum of the even numbers to 100:

$$2 + 4 + 6 + 8 + \ldots . + 100.$$

Note that the sum of the results of the last two series is equal to that of the first.

4. Find the sum of the first n numbers:

$$1 + 2 + 3 + 4 + \ldots . + n.$$

5. Find the sum of each of the progressions of n numbers:

$$1 + 4 + 7 + \ldots . + 3n - 2.$$
$$2 + 5 + 8 + \ldots . + 3n - 1.$$
$$3 + 6 + 9 + \ldots . + 3n.$$

6. If the fourth term is 14 and the seventh term 26, find
(*a*) The sum of the first seven terms.
(*b*) The sum of the terms from the fourth to the eleventh [inclusive.

7. In the progression

$$a, \quad a + d, \quad a + 2d, \ldots . \; a + (2m + 1)d,$$

find the separate sums of the alternate terms

$$a + (a + 2d) + (a + 4d) + \ldots . + (a + 2md), \text{ and}$$
$$a + d + (a + 3d) + (a + 5d) + \ldots . + [a + (2m + 1)d],$$

and take the difference of these sums.

8. In a progression of five terms the middle term is m and the common difference h. Form the sum of the terms and the sum of their squares.

9. The sum of three numbers in A.P. is 21 and the sum of their squares 197. Find the numbers.

10. Find five numbers in A.P. of which the sum is 25 and the sum of the squares 165.

REMARK. Problems like the two preceding are most readily solved by taking the middle term as the unknown quantity.

11. In an A.P. of five terms the middle term is m and the common difference h. Form the product of the first, second, fourth and fifth terms.

12. In an A.P. the product of the fourth and sixth terms exceeds the product of the third and seventh by 48. Find the common difference.

13. In an A.P. of six terms the sum of the first three terms is 15 and of the last three 51. Find the progression.

14. The sum of an A.P. of six terms is 60, and the last term is three times the first. Find the progression.

15. Find an A.P. of seven terms such that the sum of the last three terms shall exceed the sum of the first three by 36 and the product of the second and last shall be 100.

16. By substituting in the value (5) of Σ the value (1) of l, express the sum of the progression in terms of the first term, common difference and number of terms.

17. Express the sum in terms of the last term, common difference and number of terms.

18. Find the sum of six terms of the A.P. whose first term is 30 and common difference -4. Also, find the sum of ten terms and explain the equality of the answers to the two questions.

19. The first term of an A.P. is 9, the common difference -1 and the sum 35. Find the number of terms and explain the two answers.

20. Find seven numbers in A.P. such that the sum of the squares of the first and seventh shall be 234 and the sum of the squares of the third and fifth 170.

21. A pedestrian having to make a journey of 180 miles goes 40 miles the first day and diminishes his day's journey 4 miles on each succeeding day. How long will it take him to reach his destination? Explain the two answers.

22. Find the sum of 12 terms of the progression of which the first term is 22 and the common difference -4.

☞ Teachers requiring additional problems in arithmetical progression will find them in the Appendix.

SECTION II. GEOMETRICAL PROGRESSION.

286. *Def.* A **geometrical progression** consists of a series of terms each of which is formed by multiplying the term preceding by a constant factor.

An arithmetical progression is formed by continued addition or subtraction; a geometrical progression by continued multiplication or division.

Def. The factor by which each term is multiplied to form the next one is called the **common ratio.**

The common ratio is analogous to the common difference in an arithmetical progression.

In other respects the same definitions apply to both.

EXAMPLES.

$$2, \quad 6, \quad 18, \quad 54, \quad \text{etc.},$$

is a progression in which the first term is 2 and the common ratio 3.

$$2, \quad 1, \quad \tfrac{1}{2}, \quad \tfrac{1}{4}, \quad \tfrac{1}{8}, \quad \text{etc.},$$

is a progression in which the ratio is $\frac{1}{2}$.

$$+3, \quad -6, \quad +12, \quad -24, \quad \text{etc.},$$

is a progression in which the ratio is -2.

NOTE. A progression like the second one above, formed by dividing each term by the same divisor to obtain the next term, is included in the general definition, because dividing by any number is the same as multiplying by its reciprocal. Geometrical progressions may therefore be divided into two classes, increasing and decreasing. In an increasing progression the common ratio is greater than unity and the terms go on increasing; in a decreasing progression the ratio is less than unity and the terms go on diminishing.

REMARK. In a progression in which the ratio is negative the terms will be alternately positive and negative.

287. *Def.* A **geometrical mean** between two quantities is the square root of their product.

Def. The intermediate terms of a geometrical progression are called **geometrical means** between the extreme terms.

EXERCISES.

Form four terms of each of the following geometrical progressions:

1. First term, 2; common ratio, 3.
2. First term, 4; common ratio, -3.
3. First term, 1; common ratio, -1.
4. Second term, $\frac{2}{3}$; common ratio, $\frac{3}{2}$.
5. Third term, $\frac{4}{5}$; common ratio, $-\frac{1}{2}$.
6. Third term, a; common ratio, b.

Problems in Geometrical Progression.

288. In a geometrical progression, as in an arithmetical one, there are five quantities, any three of which determine the progression and enable the other two to be found. They are:

a, the first term.
r, the common ratio.
n, the number of terms.
l, the last term.
Σ, the sum of the n terms.

The general expression for the geometrical progression will be

$$a, \quad ar, \quad ar^2, \quad ar^3, \quad \text{etc., to} \quad l,$$

because each of these terms is formed by multiplying the preceding one by r.

The same problems present themselves in the two progressions. Those for the geometrical one are as follows:

289. PROBLEM I. *Given the first term, the common ratio and the number of terms, to find the last term.*

The progression will be

$$a, \quad ar, \quad ar^2, \quad \text{etc.}$$

We see that the exponent of r is less by 1 than the number of the term, and since it increases by 1 for each term added it must remain less by 1, how many terms soever we take. Hence the nth term is

$$l = ar^{n-1}. \qquad (1)$$

290. PROBLEM II. *Given the last term, the common ratio and the number of terms, to find the first term.*

The solution is found by dividing both members of (1) by r^{n-1}, which gives

$$a = \frac{l}{r^{n-1}}. \qquad (2)$$

291. PROBLEM III. *Given the first term, the last term and the number of terms, to find the common ratio.*

From (1) we find $\quad r^{n-1} = \frac{l}{a}.$

Extracting the $(n-1)$th root of each member, we have

$$r = \left(\frac{l}{a}\right)^{\frac{1}{n-1}}. \qquad (3)$$

[The solution of Problem IV. requires us to find n from equation (1), and belongs to a higher department of algebra.]

EXERCISES.

1. If the first term is a and the common ratio p^2, express the second, third, fourth and nth terms.
2. If the nth term is x and the common ratio $1+\rho$, express the first term.
3. What must be the common ratio that the first term may be x and the nth one y?

292. PROBLEM V. *To find the sum of all n terms of a geometrical progression.*

We have $\quad \Sigma = a + ar + ar^2 + \text{etc.} + ar^{n-1}.$

Multiply both sides of this equation by r. We then have

$$r\Sigma = ar + ar^2 + ar^3 + \text{etc.} \ldots\ldots + ar^n.$$

Now subtract the first of these equations from the second. It is evident that in the second equation each term of the second member is equal to that term of the second member of the first equation, which is one place farther to the right. Hence, when we subtract, all the terms will cancel each other except the first of the first equation and the last of the second.

Illustration. The following is a case in which $a = 2$, $r = 3$, $n = 6$:

$$\Sigma = 2 + 6 + 18 + 54 + 162 + 486.$$
$$3\Sigma = 6 + 18 + 54 + 162 + 486 + 1458.$$

Subtracting, $\quad 3\Sigma - \Sigma = 1458 - 2 = 1456,$

or $\quad 2\Sigma = 1456 \quad \text{and} \quad \Sigma = 728.$

Returning to the general problem, we have by subtraction

$$(r-1)\Sigma = ar^n - a = a(r^n - 1);$$

whence
$$\Sigma = a\frac{r^n - 1}{r - 1} = a\frac{1 - r^n}{1 - r}. \qquad (4)$$

It will be most convenient to use the first form when $r > 1$ and the second when $r < 1$.

By this formula we are enabled to compute the sum of the terms of a geometrical progression without actually forming all the terms and adding them.

EXERCISES.

1. A farrier having told a coachman that he would charge him \$3 for shoeing his horse, the latter objected to the price. The farrier then offered to take 1 cent for the first nail, 2 for the second, 4 for the third, and so on, doubling the amount for each nail, which offer the coachman accepted. There were 32 nails. Find how much the coachman had to pay for the last nail, and how much in all.

2. Having the progression

$$a + ar + ar^2 + ar^3 + ar^4,$$

what is the common ratio of the new progression formed by taking

(*a*) Every alternate term of this progression?

(*b*) Every *m*th term?

3. Insert two geometrical means between the extremes a and ah^6.

NOTE. See Problem III. and note that when two means are inserted we shall have a progression of four terms, and, in general, that when n means are inserted between two extremes the whole forms a G.P. of $n+2$ terms.

4. Insert three geometrical means between the extremes m and m^5.

5. Insert two geometrical means between the extremes ab and a^4b^7.

6. What is the geometrical mean of a and $a^{\frac{1}{3}}$? Of a and ab? Of $a + b$ and $a - b$?

7. Insert two geometrical means between ab and a^2b^3.

8. The arithmetical mean of two numbers is $6\frac{1}{2}$ and their geometrical mean is 6. Find the numbers.

9. Find two numbers whose sum is 30 and the sum of whose arithmetical and geometrical means is 24.

10. In the following series of numbers and expressions state which are geometrical progressions and which are not such:

(*a*) 1, 3, 6, 10, etc.
(*b*) 2, 4, 8, 16, etc.
(*c*) 4, 2, $\frac{1}{2}$, $\frac{1}{4}$, etc.
(*d*) 2, -3, $+4$, -5, etc.
(*e*) a, a^2, a^3, a^4, etc.
(*f*) a, $2a^3$, $3a^5$, $5a^7$, etc.
(*g*) $3m^4$, $-6m^6$, $12m^8$, $-24m^{10}$, etc.

11. If we take the geometrical progression

$$a, \quad ar, \quad ar^2, \quad ar^3, \quad ar^4, \quad \text{etc.}, \qquad (a)$$

and form a new series of terms by adding each term to the one next following, thus:

$$a(1+r), \quad a(r+r^2), \quad a(r^2+r^3), \quad a(r+r^4), \quad \text{etc.},$$

is this new series or is it not a geometrical progression? and if it is such, what is its common ratio?

12. If, in the preceding example, we form a new series by subtracting each term of the progression from the term next following, will the new series be a geometrical progression or will it not.

13. If we multiply all the terms of a G.P. by the same constant factor, will the products form a G.P.?

14. If we add the same constant quantity to all the terms, will the sums form a G.P.

15. Do the reciprocals of a G.P. form another G.P.? If so, what is the common ratio?

16. What is the continued product of the first three terms of the progression (*b*), Ex. 10? Of the first four terms? Of the first n terms?

17. One number exceeds another by 15, and the arithmetical mean of the two is greater than the geometrical mean by $\frac{5}{2}$. What are the numbers?

18. Find a progression of three terms of which the first term is 1 and the continued product of the three terms 343.

19. Find the common ratio of that progression of which

the sum of the third and fourth terms is one fourth the sum of the first and second.

20. What is the common ratio when the sum of the first and fourth terms is to the sum of the second and third as 13 to 5?

Note that $1 + r^3$ is divisible by $1 + r$.

☞ Teachers requiring additional problems in geometrical progression will find them in the Appendix.

Limit of the Sum of a Progression.

293. THEOREM. *If the absolute value of the common ratio of a geometrical progression is less than unity, then there always will be a certain quantity which the sum of all the terms can never exceed, no matter how many terms we take.*

EXAMPLE 1. The sum of the progression

$$\tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \text{etc.},$$

in which the common ratio is $\frac{1}{2}$, can never amount to 1, no matter how many terms we take. To show this, suppose that one person owed another a dollar and proceeded to pay him a series of fractions of a dollar in geometrical progression, namely,

$$\tfrac{1}{2},\quad \tfrac{1}{4},\quad \tfrac{1}{8},\quad \tfrac{1}{16},\quad \text{etc.}$$

When he paid him the $\frac{1}{2}$ he would still owe another $\frac{1}{2}$, when he paid the $\frac{1}{4}$ he would still owe another $\frac{1}{4}$, and so on. That is, at every payment he would discharge one half the remaining debt. Now there are two propositions to be understood in reference to this subject:

I. *The entire debt can never be discharged by such payments.*

For, since the debt is *halved* at every payment, if there was any payment which discharged the whole remaining debt, the half of a thing would be equal to the whole of it, which is impossible.

II. *The debt can be reduced below any assignable limit by continuing to pay half of it.*

For, however small the debt may be made, another payment will make it smaller by one half; hence there is no smallest amount below which it cannot be reduced.

These two propositions, which seem to oppose each other, hold the truth between them, as it were. They constantly enter into the higher mathematics, and should be well understood. We therefore present another illustration of the same subject.

Ex. 2. Suppose AB to be a line of given length. Let us go one half the distance from A to B at one step, one fourth at the second, one eighth at the third, etc. It is evident that at each step we go half the distance which remains. Hence the two principles just cited apply to this case. That is:

(1) We can never reach B by a series of such steps, because we shall always have a distance equal to the last step left.

(2) But we can come as near B as we please, because every step carries us over half the remaining distance.

This result is often expressed by saying that we should reach B by taking an infinite number of steps. This is a convenient form of expression, and we may sometimes use it; but it is not logically exact, because no conceivable number can be really infinite. The assumption that infinity is an algebraic quantity often leads to ambiguities and difficulties in the application of mathematics.

294. *Def.* The **limit** of the sum Σ of a geometrical progression is a quantity which Σ may approach so that its difference shall be less than any quantity we choose to assign, but which Σ can never reach.

EXAMPLES. 1. The limit of the sum

$$\tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{16} + \text{etc.}$$

is 1, because this sum fulfils the two conditions

(a) That it can never be as great as 1;

(b) That it can be brought as near to 1 as we please by increasing the number of terms added.

2. The point B in the preceding figure is the limit of all the steps that can be taken in the manner described.

The following principle will enable us to find the limit of the sum of a progression:

295. *Principle.* If $r < 1$, the power r^n can be made as small as we please by increasing the value of n, but can never be made equal to 0.

Suppose, for instance, that

$$r = \tfrac{3}{4} = 1 - \tfrac{1}{4}.$$

Then every time we multiply by r we diminish r^n by $\frac{1}{4}$ of its former value; that is,

$$\begin{aligned} r^2 &= \tfrac{3}{4}r = (1 - \tfrac{1}{4})r = r - \tfrac{1}{4}r, \\ r^3 &= \tfrac{3}{4}r^2 = r^2 - \tfrac{1}{4}r^2, \\ r^4 &= \tfrac{3}{4}r^3 = r^3 - \tfrac{1}{4}r^3, \\ &\text{etc.} \quad \text{etc.} \quad \text{etc.} \end{aligned}$$

296. Problem. *To find the limit of the sum of a geometrical progression.*

To solve this problem we must take the expression for the sum of n terms, and see to what limit it approaches when we suppose n to increase indefinitely. The required expression, as found in § 292, is

$$\Sigma = a\frac{1 - r^n}{1 - r},$$

which we may put into the form

$$\Sigma = \frac{a}{1-r} - \frac{a}{1-r}r^n. \qquad (6)$$

This expression for Σ, the sum of n terms, is identically the same as (4), but different in form.

If r is greater than unity, the quantity r^n will increase indefinitely when n increases indefinitely, and the expression will have no limit.

If r is less than unity, then, by § 295, when n increases indefinitely r^n will approach 0 as its limit, and therefore the expression $\dfrac{a}{1-r}\,r^n$ will also approach 0, so that we shall have

$$\text{Limit of } \Sigma = \frac{a}{1-r}. \qquad (7)$$

Example 1. If $a = 1$ and $r = \frac{1}{2}$, then $\dfrac{a}{1-r} = 2$; and

When $n = 2$, $1 + \frac{1}{2} = 2 - \frac{1}{2}$;
When $n = 3$, $1 + \frac{1}{2} + \frac{1}{4} = 2 - \frac{1}{4}$;
When $n = 4$, $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = 2 - \frac{1}{8}$;
When $n = 5$, $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = 2 - \frac{1}{16}$;
etc. etc. etc.

We see that as we make n equal to 5, 6, 7, etc., indefinitely, the sum of the series is less than 2 by $\frac{1}{16}$, $\frac{1}{32}$, $\frac{1}{64}$, etc., indefinitely.

Since, by continually halving a quantity, the parts can be made as small as we please, the limit of the sum of the series is 2.

Ex. 2. $\qquad a = 1; \quad r = \frac{1}{3}.$

We now have

$$\frac{a}{1-r} = \frac{3}{2};$$

and

$$\begin{array}{ll}
\text{For } n = 2, & 1 + \frac{1}{3} = \frac{3}{2} - \frac{3}{18}; \\
\text{For } n = 3, & 1 + \frac{1}{3} + \frac{1}{9} = \frac{3}{2} - \frac{3}{54}; \\
\text{For } n = 4, & 1 + \frac{1}{2} + \frac{1}{9} + \frac{1}{27} = \frac{3}{2} - \frac{3}{162}; \\
\quad \text{etc.} & \qquad \text{etc.} \qquad\qquad \text{etc.}
\end{array}$$

We see how as n increases the sum approaches $\frac{3}{2}$ as its limit.

Ex. 3. $\qquad a = 1, \quad r = -\frac{1}{2}.$

We then have

$$\frac{a}{1-r} = \frac{2}{3};$$

and

$$\begin{array}{ll}
\text{For } n = 2, & 1 - \frac{1}{2} = \frac{2}{3} - \frac{2}{12}; \\
\text{For } n = 3, & 1 - \frac{1}{2} + \frac{1}{4} = \frac{2}{3} + \frac{2}{24}; \\
\text{For } n = 4, & 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} = \frac{2}{3} - \frac{2}{48}; \\
\text{For } n = 5, & 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} = \frac{2}{3} + \frac{2}{96}; \\
\quad \text{etc.} & \qquad \text{etc.} \qquad\qquad \text{etc.}
\end{array}$$

We see that as n increases the sums are alternately greater and less than the limit $\frac{2}{3}$, but that they continue to approach it as before.

EXERCISES.

Find the limits of the sums of the following progressions:

1. $\frac{1}{4} + \frac{1}{16} + \frac{1}{64} +$ etc., *ad infinitum.*

2. $\frac{1}{4} - \frac{1}{16} + \frac{1}{64} -$ etc., *ad infinitum.*

3. $\frac{1}{(1+S)} + \frac{1}{(1+S)^2} + \frac{1}{(1+S)^3} +$ etc., *ad infinitum.*

4. $\frac{1}{S-1} - \frac{1}{(S-1)^2} + \frac{1}{(S-1)^3} -$ etc., *ad infinitum.*

5. $\frac{3}{4} + \frac{3^2}{4^2} + \frac{3^3}{4^3} +$ etc., *ad infinitum.*

6. $\frac{4}{5} - \frac{4^2}{5^2} + \frac{4^3}{5^3} -$ etc., *ad infinitum.*

7. $\frac{m}{1+m} + \frac{m^2}{(1+m)^2} + \frac{m^3}{(1+m)^3} +$ etc., *ad infinitum.*

8. $\frac{m}{1+m} - \frac{m^2}{(1+m)^2} + \frac{m^3}{(1+m)^3} -$ etc., *ad infinitum.*

9. From a cistern of water half is drawn into tub A, half of what is left into tub B, half of what is then left into tub A, and so on alternately. If this were continued without limit, what proportion of the water would go into the respective tubs?

10. A man starts from a point A towards a point B one mile distant. But he stops at a point b two thirds of the way from A to B; then walks to a point c two thirds of the way from b to A; then to a point d two thirds of the way from c to b, and so on alternately, going in each direction two thirds of the way back to the point from which he last set out. To what point would he continually approach, and what is the limit of the distance he could ever walk?

Begin by drawing a line to represent the distance from A to B, and mark the points b, c, d, etc., upon it.

CHAPTER II.

VARIATION.

297. *Def.* When the value of one quantity depends upon that of another, the one quantity is called a **function** of the other.

EXAMPLE 1. The *time* required for a train to perform a journey depends upon the *speed* of the train. Hence in this case the time is a function of the speed.

Ex. 2. The value of a chest of tea depends upon its weight. Hence the value is a function of the weight.

Ex. 3. The weight which a man can carry is a function of his strength.

Let the student, as an exercise, name other cases in which one quantity is a function of another.

298. When one quantity u is a function of a second quantity x, then for every value of x there will be a *corresponding* value of u.

EXAMPLE 1. If tea is worth 50 cents a pound, then to the quantity

1 pound will correspond the value 50 cents;
2 pounds " " " " $1.00;
3 " " " " " $1.50;
etc. etc. etc.

Ex. 2. If a train has to make a journey of 60 miles at a uniform speed, then to the speed

60 miles an hour will correspond the time 60 minutes;
45 " " " " " " " 80 "
40 " " " " " " " 90 "
30 " " " " " " " 120 "
etc. etc. etc.

299 *Direct Variation. Def.* When two quantities are so related that any two values of the one have the same ratio as the corresponding values of the other, the one is said to **vary directly** as the other.

When one quantity varies directly as another, then by doubling the one we double the other, by trebling the one we treble the other, by halving the one we halve the other, etc.

EXAMPLE. The weight of an article varies directly as the quantity. When we halve, double or treble the quantity, the weight will also be halved, doubled or trebled.

NOTE. In speaking of direct variation we may omit the word *direct.*

300. *Expression of the fact that one quantity y varies directly as another quantity x.* Let us put

$a \equiv$ any one value of y;

$c \equiv$ the corresponding value of x;

then, by definition, we must have, for all values of x and y,

$$x : a = y : c.$$

This proportion gives

$$ay = cx,$$

whence

$$y = \frac{c}{a}x.$$

Here we may regard $\frac{c}{a}$ as a factor by which we multiply any value of x to obtain the corresponding value of y. Hence

The fact that one quantity varies as another is expressed by equating the one to the product of the other into a constant factor.

EXAMPLE. Given that y varies as x, and that

when $\qquad x = 3,$

then $\qquad y = 5;$

it is required to express y in terms of x.

Solution. By definition we have, for all values of x and y:

$$y : 5 = x : 3;$$

$$\therefore \quad 3y = 5x;$$

$$\therefore \quad y = \tfrac{5}{3}x,$$

which is the required expression.

EXERCISES.

1. Given that p varies as r, and that when $r = 3$, then $p = 2$; it is required to express p in terms of r, and to find the values of p corresponding to $r = 1, 2, 5, 9, 12, 16$ and 33 respectively.

2. If the value of a quantity a of iron is represented by b, what will represent the value of a quantity x of iron?

3. Given that y varies as x, and that

when $\quad x = a,$

then $\quad y = b;$

it is required to express the values of y when

$$\begin{array}{ll} (a) & x = 3a; \\ (b) & x = 3a + m; \\ (c) & x = 3a + 2m; \\ (d) & x = 3a + 3m. \end{array}$$

4. Prove that if y varies as x, and that if we suppose x to take a series of values in arithmetical progression, such as

$$\begin{array}{l} x = a, \\ x = a + d, \\ x = a + 2d, \\ x = a + 3d, \\ \text{etc.} \quad \text{etc.}, \end{array}$$

then the corresponding values of y will also form an arithmetical progression.

5. Prove that if y varies as x, the difference between any two values of y will have the same ratio to the difference between the corresponding values of x which any value of y has to the corresponding value of x.

301. *Inverse Variation.* One quantity is said to **vary inversely** as another when the ratio of any two values of the one is the inverse ratio of the corresponding values of the other.

If $\quad y_1$ corresponds to x_1

and $\quad y_2$ " " x_2,

then, when y varies inversely as x,

$$y_1 : y_2 = x_2 : x_1.$$

Therefore $$y_2 x_2 = y_1 x_1.$$

Hence, because x_1 and y_1 as well as x_2 and y_2 may be any pair of values of the quantities,

In inverse variation the product of two corresponding values of the quantities is always the same.

Hence, also, in inverse variation

One of the quantities can always be found by dividing some dividend by the other quantity.

This dividend is the product of any two corresponding values of the quantities.

EXAMPLE. If $y = 4$ when $x = 2$, and y varies inversely as x, then

To	$x = 2$	corresponds	$y = 4$;
"	$x = 4$	"	$y = 2$;
"	$x = 8$	"	$y = 1$;
"	$x = 16$	"	$y = \frac{1}{2}$;
"	$x = 32$	"	$y = \frac{1}{4}$;
	etc.		etc.

It will be seen that each product of a value of x into the corresponding value of y is 8.

EXERCISES.

1. If y varies inversely as x, and when $x = 3$, $y = 8$, it is required to make a little table showing the values of y for $x = 1, 2, 3$, etc., to 12.

2. If y varies inversely as x, and when $x = a$, then $y = b$, it is required to express the values of y for $x = 1$, $x = 2$ and $x = 3$.

3. Given that u varies inversely as z, and that when $z = 3$ the corresponding value of u is greater by 5 than it is when $z = 4$; it is required to express the relation between u and z by an equation.

NOTE. We may solve this problem with most elegance by taking for the unknown quantity the product of any value of u by the corresponding value of z. Let p represent this product, and let u_2 represent the value of u when $z = 4$. We then have the two equations

$$4u_2 = p,$$
$$3(u_2 + 5) = p.$$

Eliminating u_2, we find $p = 60$. Hence the required equation is

$$u = \frac{p}{z} = \frac{60}{z},$$
$$uz = 60.$$

4. Given that u varies inversely as z, and that when $z = \frac{1}{2}$ the value of u is greater by 1 than it is when $z = 1$; it is required to express the relation between u and z.

5. Given that u varies inversely as z, and that the sum of the values of u for $z = 1$ and $z = 2$ is 5; it is required to express the relation between u and z by an algebraic equation.

6. Show that the quantity of goods which can be purchased with a given sum of money varies inversely as the price.

7. Show that the time required to perform a journey varies inversely as the speed.

302. *Def.* One quantity is said to vary as the square of another when any two values of the one have the same ratio as the squares of the corresponding values of the other.

If u and z and u_1 and z_1 are pairs of corresponding values, then when u varies as the square of z, we must always have

$$u : u_1 = z^2 : z_1^2.$$

In the same way as in direct variation (§ 300) may be shown:

When one quantity varies as the square of another it is equal to that square multiplied by some constant factor.

303. *Def.* One quantity is said to vary inversely as the square of another when the ratio of any two values of the one is the inverse ratio of the squares of the two corresponding values of the other.

If u varies inversely as the square of z, and if u_1 is the value of u when $z = z_1$, then we always have

$$u : u_1 = z_1^2 : z^2,$$

which gives

$$u = \frac{u_1 z_1^2}{z^2}.$$

We therefore conclude:

When one quantity varies inversely as the square of another it is equal to some constant quantity divided by the square of that other.

EXAMPLES AND EXERCISES.

1. It is found that the attraction between two bodies varies inversely as the square of their distance apart. If when the distance is 1 foot the attraction is represented by 2, it is required to express the attraction at the distances 2, 3, 4, 5 and 10 feet.

2. Supposing the moon to be 60 radii of the earth (that is, 60 times as far from the earth's centre as any point on the earth's surface), what would a ton of coal weigh at the distance of the moon?

NOTE. Because the weight of the coal is equal to the attraction of the earth, it varies inversely as the square of its distance from the earth's centre.

3. The area of a circle varies as the square of its diameter. If the diameter of one circle is $2\frac{1}{2}$ times that of another, what is the ratio of their areas?

4. The apparent brilliancy of a candle, that is, the amount of light which it will cast upon a small surface held perpendicular to the line from the candle to the surface, varies inversely as the square of the distance. If we represent by 1 the brilliancy of the candle at a distance of 8 feet, what numbers will represent its brilliancy at the distances of 1 inch, 1 foot, 4 feet and 16 feet respectively?

5. The volume of a sphere varies as the cube of its diameter. If one sphere has 3 times the diameter of another, what is the ratio of their volumes?

6. If the diameters of two spheres are to each other as $5 : 2$, what is the ratio of their volumes?

304. *Combined Variation.* The value of one quantity may depend upon the values of several other quantities. The first quantity is then called the **function**, and the others the **independent variables.**

We may then suppose all the independent variables but one to be constant, and the remaining one to vary, and express the law of variation of the function.

Having thus supposed each of the independent variables in succession to vary, the law of variation of the function is

expressed by stating how it varies with respect to each independent variable.

EXAMPLE 1. The *time* required to perform a journey is a function of the *speed* and of the *distance.*

If we suppose the distance to remain constant and the *speed* to vary, the *time* will vary inversely as the *speed.*

If we suppose the speed to remain constant and the *distance* to vary, the *time* will vary directly as the *distance.*

These two relations are expressed by saying that the time varies directly as the distance and inversely as the speed.

Ex. 2. Suppose a candle at C to illuminate a surface at I.

$C \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; I$

If we suppose the candle to become 2, 3, 4 or n times as bright, the distance CI remaining constant, the illumination of I will become 2, 3, 4 or n times as great.

If we suppose that while the brightness of the candle remains constant the distance CI becomes 2, 3 or n times as great, the illumination will be reduced to $\frac{1}{4}$, $\frac{1}{9}$ or $\frac{1}{n^2}$ of its first amount.

These two facts are expressed by saying that the illumination varies directly as the brightness of the candle and inversely as the square of the distance.

305. *Expression of Combined Variation.* If a function u varies directly as the quantities p, q, r, etc., then, by § 300, its expression must contain each of these quantities as a simple factor.

If it varies inversely as the quantities u, v, w, etc., then, by § 301, its expression must contain each of these quantities as a divisor.

Therefore the fact that u varies directly as $p, q, r \ldots$ and inversely as $u, v, w \ldots$ is expressed by an equation of the form

$$u = \frac{Apqr \ldots}{uvw \ldots},$$

in which A represents some constant quantity, the value of which must be chosen so as to fulfil the conditions of each special problem.

EXERCISES.

1. Express that the function u varies directly as the square of m and inversely as the cube of x. Ans. $u = \frac{Am^2}{x^3}$.

2. Express that the function s varies directly as m and n and inversely as the square of r.

3. Express that the function q varies directly as m and as the square of x, and inversely as the cube of h and as the fourth power of z.

4. If we represent by unity the brilliancy with which a candle of brightness unity illuminates a page 4 feet away, what number will represent the illumination of a page 8 feet away by a candle of brightness 3?

5. If an electric light gives as much light as 2500 candles, at what distance will it illuminate the page of a book as brightly as a candle 5 feet away will?

6. At C is a candle, and at E is an electric light equal to 1600 candles, distant 500 feet from C. It is required to find the distances from the candle to the two points X and Y from which the candle and the electric light appear equally brilliant.

7. If of two lights a feet apart the brighter gives r times as much light as the fainter, it is required to find the positions of the two points on the straight line joining them from which the two lights appear equally bright.

8. An electric light 80 feet distant was found to throw as much light on a book as a candle 2 feet distant. If another candle twice as bright is used, how far must the electric light be placed to give as much light as this brighter candle does at 2 feet distance?

CHAPTER III.

LOGARITHMS.

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306. To every number corresponds a certain other number called its *common logarithm.*

Def. The **common logarithm** of a number is the exponent with which 10 must be affected in order to produce the number.

The term *common logarithm* is used because there are other logarithms than the common ones. Since we are at present only concerned with the latter, we shall drop the adjective common.

To express the logarithm of a number n we write log n, so that

$$\log n \equiv \text{the logarithm of } n.$$

EXAMPLES.

$$\left.\begin{array}{lll} 10^0 = 1; & \therefore \log 1 = & 0. \\ 10^1 = 10; & \therefore \log 10 = & 1. \\ 10^2 = 100; & \therefore \log 100 = & 2. \\ 10^{-2} = \frac{1}{100}; & \therefore \log \frac{1}{100} = & -2. \\ \quad\text{etc.} & \qquad\quad \text{etc.} & \end{array}\right\} \qquad (a)$$

Thus we have

THEOREM I. *The logarithm of* 1 *is zero.*
The logarithm of 10 *is* 1.
etc. etc.

307. If we call any number x, and put

$$y = \log x,$$

we have, by definition,

$$10^y = x. \qquad (b)$$

If we suppose y to increase from 0 to 1, x will increase from 1 to 10. Hence

THEOREM II. *The logarithm of a number betwen* 1 *and* 10 *is a positive fraction between* 0 *and* + 1.

308. If we multiply the equation (b) by 10, we have

$$10^{y+1} = 10x;$$

that is,

$$\log 10x = y + 1.$$

Hence

THEOREM III. *Every time we multiply a number by* 10 *we increase its logarithm by unity.*

309. By dividing (b) by 10 we obtain

THEOREM IV. *Every time we divide a number by* 10 *we diminish its logarithm by unity.*

310. In order that 10^y may be less than unity, y must be negative. Since we have

$$\begin{aligned} 10^{-1} &= .1, \\ 10^{-2} &= .01, \\ 10^{-3} &= .001, \\ 10^{-4} &= .0001, \\ \text{etc.} &\quad \text{etc.}, \end{aligned}$$

we see that as we increase the exponent negatively the number diminishes without limit. Hence

THEOREM V. *The logarithm of a proper fraction is negative.*

THEOREM VI. *The logarithm of* 0 *is negative infinity.*

311. The use of logarithms is founded on the four following theorems.

THEOREM VII. *The logarithm of a product is equal to the sum of the logarithms of its factors.*

Proof. Let p and q be two factors, and suppose

$$h = \log p, \qquad k = \log q.$$

Then $\quad 10^h = p, \qquad 10^k = q.$

Multiplying, $\quad 10^h 10^k = 10^{h+k} = pq.$

Whence, by definition,

$$h + k = \log (pq),$$

or $\quad \log p + \log q = \log (pq).$

This proof may be extended to any number of factors.

THEOREM VIII. *The logarithm of a quotient is found by subtracting the logarithm of the divisor from that of the dividend.*

Proof. Dividing instead of multiplying the equations in the last theorem, we have

$$\frac{10^h}{10^k} = 10^{h-k} = \frac{p}{q}.$$

Hence, by definition, $$h - k = \log\frac{p}{q},$$

or $$\log p - \log q = \log\frac{p}{q}.$$

THEOREM IX. *The logarithm of any power of a number is equal to the logarithm of the number multiplied by the exponent of the power.*

Proof. Let $h = \log p$, and let n be the exponent.

Then $$10^h = p.$$

Raising both sides to the nth power,

$$10^{nh} = p^n.$$

Whence $$nh = \log p^n,$$

or $$n \log p = \log p^n.$$

THEOREM X. *The logarithm of a root of a number is equal to the logarithm of the number divided by the index of the root.*

Proof. Let s be the number, and let p be its nth root, so that

$$p = \sqrt[n]{s} \quad \text{and} \quad s = p^n.$$

Hence $$\log s = \log p^n = n \log p. \qquad \text{(Th. IX.)}$$

Therefore $$\log p = \frac{\log s}{n},$$

or $$\log \sqrt[n]{s} = \frac{\log s}{n}.$$

EXERCISES.

Express the following logarithms in terms of $\log p$, $\log q$, $\log x$, $\log y$, $\log (x-y)$ and $\log (x+y)$:

1. $\log py$. Ans. $\log p + \log y$.
2. $\log qx$
3. $\log pqy$.
4. $\log p(x+y)$.
5. $\log x^2 + xy$. Ans. $\log x + \log (x+y)$.
6. $\log 10\,p$. Ans. $\log p + 1$.
7. $\log 10xy$.
8. $\log p^2x$. Ans. $2 \log p + \log x$.
9. $\log p^3x$.
10. $\log p^nx$.
11. $\log 10p^2x^2$.
12. $\log 10p^mx^n$.
13. $\log 10^n$.
14. $\log \dfrac{1}{10^n}$.
15. $\log \dfrac{x}{q}$.
16. $\log \dfrac{y}{q^2}$.
17. $\log \dfrac{y^2}{q}$.
18. $\log \dfrac{y^2}{q^3}$.
19. $\log \dfrac{xy}{pq}$.
20. $\log \dfrac{10xy}{pq}$.
21. $\log \dfrac{xy}{10pq}$.
22. $\log \dfrac{xy}{100pq}$.
23. $\log \sqrt{x}$.
24. $\log \sqrt{(x+y)}$.
25. $\log p^{\frac{1}{2}}y^{\frac{3}{2}}$.
26. $\log \sqrt{10}$.
27. $\log \sqrt[3]{10}$.
28. $\log \sqrt{pq}$.
29. $\log \sqrt{10(x+y)}$.
30. $\log \sqrt{\dfrac{xy}{10}}$.
31. $\log (x^2 - y^2)$.
32. $\log (x^2 - xy)$.
33. $\log (x^4 - x^2y^2)$.
34. $\log p(x^4 - x^2y^2)$.
35. $\log (x^2 - y^2)^{\frac{1}{3}}$.
36. $\log p^{\frac{1}{3}}q^{\frac{1}{2}}(x^4 - x^2y^2)^{\frac{1}{2}}$.

Table of Logarithms.*

312. The logarithm of a number consists of an integer and a decimal fraction.

The decimal fraction is called the **mantissa** of the logarithm.

The integer is called the **characteristic** of the logarithm.

A table of logarithms gives only the mantissæ.

313. *To find the mantissa of the logarithm of a given number.*

CASE I. When the number has three or fewer figures.

RULE. *Find in the left-hand column that line which contains the first two figures of the number, and select that column which has the third figure at its top. The four figures in this column on the line selected form the mantissa of the logarithm.*

EXAMPLES. Mantissa of log 134 = .1271;
" " log 17 = .2304;
" " log 707 = .8494.

CASE II. When the number has four or more figures.

RULE. *Find the mantissa of the first three figures as in Case I., and call this mantissa m.*

Take the difference D between this mantissa and that next following it in the table.

Imagine the third figure of the given number to be followed by a decimal point, and multiply D by the fourth and fifth figures considered as decimals.

Add the product to m; the sum will be the logarithm required.

EXAMPLE. To find mantissa of log 17762.

$$\begin{aligned} \text{mant. of log } 177 &= .2480 \\ D = 24;\ .62 \times 24 &= \underline{\quad 15} \\ \text{mant. log } 17762 &= .2495 \end{aligned}$$

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* It is very desirable that the pupil should learn to use logarithms in multiplication and division at the earliest period possible in his studies. The precepts for using this table are therefore arranged so that they may be used before understanding the entire theory of logarithms.

314. *Characteristics.* The characteristic of a logarithm is less by unity than the number of figures preceding the decimal point in the number to which it corresponds.

EXAMPLES. The mantissa last found gives the following logarithms:

$$\log\ 17762 = 4.2495;$$
$$\log 1776.2 = 3.2495;$$
$$\log 177.62 = 2.2495;$$
$$\log 1.7762 = 0.2495.$$

Every time we move the decimal point one place toward the left we divide the number by 10 and diminish the logarithm by unity, which leaves the mantissa unchanged (§309).

If the number is a decimal fraction, the characteristic is

-1 when the figure following the decimal point is different from zero;

-2 when one zero follows the decimal point;

-3 when two zeros follow it, and so on.

In these cases the minus sign is written above the characteristic to show that it belongs to the characteristic alone and not to the mantissa.

EXAMPLES.
$$\log 0.17762 = \bar{1}.2495;$$
$$\log 0.017762 = \bar{2}.2495;$$
$$\log 0.0017762 = \bar{3}.2495:$$
etc. etc.

EXERCISES.

Find the logarithms of the following numbers:

1. 701.	6. 7032.	11. 1.0246.
2. 700.	7. 70.32.	12. 2.0324.
3. 70.	8. 0.7032.	13. 51.623.
4. 7.	9. 0.007032.	14. 11.111.
5. 0.7.	10. 2956.	15. 3.243.

315. To find the number corresponding to a given logarithm.

RULE. 1. *Find in the table, if possible, the mantissa of the given logarithm. The corresponding figures in the left-hand column and at the top of the column are the figures of the required number.*

2. *If the mantissa is not found in the table, find the next smaller mantissa. The three corresponding figures are the first three figures of the required number.*

3. *Take the excess of the given mantissa over the next smaller one in the table and divide it by the difference between the two consecutive mantissæ in the table.*

4. *The result is a decimal fraction to be written after the three figures already found, the decimal point being dropped.*

5. *Having thus found the figures of the number, insert the decimal point corresponding to the given characteristic according to the rule of* § 314.

EXAMPLE 1. Find the number whose logarithm is 4.7267.

We find in the tables that to the mantissa .7267 correspond the figures 533. The characteristic being 4, there must be five figures before the decimal point, so that we have

$$\text{Number} = 53300.$$

Ex. 2. Find the number whose logarithm is 1.2491.

We find from the tables

Next smaller mantissa,	2480;	$N = 178.$
Given mantissa,	2491	
Excess of given mantissa,	11	

Difference of mantissæ in table = 2504 − 2480 = 24.

Figures to be added to $N = \frac{11}{24} = .46\ldots\ldots$

Therefore the figures of the number are 17846. The characteristic being 1, there are two figures before the decimal point, so that

$$\text{Number} = 17.846.$$

EXERCISES.

Find the numbers corresponding to the following logarithms:

1. 1.2032.
2. 0.7916.
3. 0.0212.
4. $\bar{1}.0212.$
5. $\bar{2}.0212.$
6. 4.0343.
7. 3.1922.
8. 2.8282.
9. 1.0602.
10. 0.9293.

No.	0	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396

No.	0	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996

APPENDIX.

Supplementary Exercises.

135. Factoring.

13. $2 - 3x^2 + x^4$.
14. $81 + 18y^2 + y^4$.
15. $x^2 + (m + n)x + mn$.
16. $x^2 - (m + n)x + mn$.
17. $ay^2 - 3ay + 2a$.
18. $mx^2 - 13mx + 40m$.
19. $m^2x^2 - 8mx + 15$.
20. $b^3x^2 - 7b^2x + 10b$.
21. $m^2y + 2my - 8y$.
22. $n^6y^3 - 3n^3y^2 - 40y$.
23. $x^4 + (2a - 3b)x^3 - 6abx^2$.
24. $x^{2n} - 4x^n - 12$.
25. $by^3 + 6by^2 + 8by$.
26. $a^4x^4 + 8a^3x^3 + 15a^2x^2$.
27. $\frac{m^3}{n^3} + 3\frac{m^2}{n^2} + 2\frac{m}{n}$.
28. $3y^2 + 18y + 24$.
29. $\frac{1}{n^2} + \frac{5}{n} + 6$.
30. $\frac{c^3}{h} + 4\frac{c^2}{h} + 3\frac{c}{h}$.
31. $2x^2 + 10x + 8$.
32. $\frac{1}{m^2} + \frac{2}{m} - 8$.
33. $\frac{2}{c^2} - \frac{10}{c} + 12$.
34. $\frac{2p^2}{q^2} + \frac{14p}{q} + 20$.
35. $\frac{a^2}{b^2} - \frac{b^2}{a^2}$.
36. $\frac{a^2x^2}{c} - \frac{b^2y^2}{c}$.
37. $\frac{m^2r^4}{n^2} - \frac{n^2s^4}{m^2}$.
38. $\frac{(a - b)^2}{(a + b)^2} - \frac{(a + b)^2}{(a - b)^2}$.
39. $\frac{a^4}{x^2} + \frac{4a^3}{x^3} + \frac{4a^2}{x^4}$.
40. $\frac{z^6}{c^3} - \frac{8z^4}{c^2} + \frac{16z^2}{c}$.
41. $c^{2n} + c^n + \frac{1}{4}$.
42. $16a^3 + a^2z + \frac{az^2}{64}$.

135a. *Factors of Quadrinomials.* Sometimes an expression containing four terms may be expressed as the product of two binomials.

EXAMPLES.

$$\begin{aligned} ax - ay + bx - by &= a(x-y) + b(x-y) \\ &= (a+b)(x-y); \\ a^2 - ar - at + rt &= a(a-r) - t(a-r) \\ &= (a-t)(a-r). \end{aligned}$$

EXERCISES.

Factor:

1. $ab + ac + mb + mc.$
2. $a^2 + am + an + mn.$
3. $cx - cy - x^2 + xy.$
4. $m^2r - m^2t - p^2r + p^2t.$
5. $a^3 - a^2q - ap + pq.$
6. $ab - bp^2 + aq^2 - p^2q^2.$
7. $ax^2 + axy + bxy + by^2.$
8. $(a^2 + b^2)xy - ab(x^2 + y^2).$
9. $(a^2 - b^2)pq - ab(p^2 - q^2).$
10. $mn(x^2 - y^2) + xy(n^2 - m^2xy).$
11. $a^3 + p^3 - ap - a^2p^2.$
12. $4ab + 2ay - 2bx - xy.$
13. $a^2 + ab(x-y) - b^2xy.$
14. $a^2 - a^2my + a^2mx - am^2xy.$
15. $1 + m + \frac{1}{m} + 1.$
16. $2 + \frac{m}{n} + \frac{n}{m} + mn + 1.$
17. $mn + 1 + 1 + \frac{1}{mn}.$
18. $2 + am + \frac{1}{am}.$
19. $2m + m^2n + \frac{1}{n}.$
20. $m^2 + \frac{m}{n} + mn + 1.$
21. $\frac{1}{mp} + \frac{1}{mq} + \frac{1}{np} + \frac{1}{nq}.$
22. $\frac{1}{mp} - \frac{1}{mq} - \frac{1}{np} + \frac{1}{nq}.$
23. $\frac{a}{m} + \frac{a}{n} - \frac{b}{m} - \frac{b}{n}.$
24. $\frac{x^2}{ab} + \frac{xy}{a^2} + \frac{xy}{b^2} + \frac{y^2}{ab}.$
25. $\frac{x^2 + y^2}{ab} - xy\left(\frac{1}{a^2} + \frac{1}{b^2}\right).$
26. $\frac{x^2 + xy^2}{ab^2} + x^2y\left(\frac{1}{a^2b} + \frac{1}{b^2}\right).$

MISCELLANEOUS EXERCISES.

Factor:

1. $\frac{4}{n^2} - \frac{9}{m^2}.$
2. $\frac{a^2}{b^2} - \frac{b^2}{a^2}.$
3. $\frac{p^2}{4} - \frac{p}{3q} + \frac{1}{9q^2}.$
4. $8a^2 + 24ax + 18x^2.$
5. $\frac{a^2}{8} - \frac{ax}{6} + \frac{x^2}{18}.$
6. $\frac{y^2}{c^2} - 6 + \frac{9c^2}{y^2}.$
7. $a^m - 4a^{2m} + 4a^{3m}.$
8. $3z + 18z^2 + 27z^3.$

9. $\frac{x^2+4y^2}{mn}+2xy\left(\frac{1}{m^2}+\frac{1}{n^2}\right)$. 10. $\frac{z^2}{9}-\frac{9}{z^2}$.

11. $\frac{x^3}{8}+\frac{a^3}{27}$. 12. $\frac{x^3}{16}-\frac{y^8}{256}$.

13. $\left(\frac{a}{x}+\frac{x}{a}\right)^2-\left(\frac{a}{x}-\frac{x}{a}\right)^2$. 14. $\frac{x^2+9y^2}{ab}-\frac{3xy}{a^2}-\frac{3xy}{b^2}$.

15. $\frac{4c^2}{9g^2}-\left(\frac{c}{g}+\frac{g}{c}\right)^2$. 16. $\frac{r^2}{t^2}+\frac{4r}{t}-21$.

17. $16\frac{a^5}{m^5}-\frac{a}{m}$. 18. $\frac{m^3}{n^3}-9\frac{mx^2}{ny^2}$.

19. $9a^2b^6+30ab^3c+25c^2$. 20. $2y^2-4yz+2z^2$.

21. $4+8x^2+4x^4$. 22. $3y^2+12yz+12z^2$.

23. $\frac{1}{(x+y)^2}-\frac{1}{(x-y)^2}$. 24. $\frac{a^2}{x^2}-\frac{x^2}{a^2}$.

25. $\frac{3}{g^2}-\frac{30}{gh}+\frac{75}{h^2}$. 26. $\frac{1}{m^2}+\frac{6}{m}+5$.

27. $\frac{1}{n^2}+\frac{2}{n}+\frac{5}{9}$. 28. $\frac{1}{n^2}-\frac{4}{3n}-\frac{5}{9}$.

29. $\frac{(a+b)^2}{(a-b)^2}-\frac{(a-b)^2}{(a+b)^2}$. 30. $\frac{m^2}{n^2}+\frac{4n^2}{m^2}+4$.

31. n^7+1. 32. n^8-1.

33. $n^{12}-1$. 34. $n^{32}-1$.

35. $1-n^{32}$. 36. $m^{32}-n^{32}$.

37. $1+u^2$. 38. $1+u^6$.

39. $a^2-2ab+b^2+a-b$. 40. $a^2+2ab+b^2-a-b$.

41. $x^2+2+\frac{1}{x^2}+x+\frac{1}{x}$. 42. $m^2-2mn+n^2-p^2$.

43. $m^2-\frac{1}{m^2}+m+\frac{1}{m}$. 44. $\frac{4}{n^2}-n^2-\frac{2}{n}+n$.

By what binomial factors must we multiply the following expressions that the products may be binomials?

(*a*) $x-a$. (*b*) x^2+ax+a^2.

(*c*) n^2+n+1. (*d*) m^3+m^2+m+1.

(*e*) x^2-ax+a^2. (*f*) $x^3-ax^2+a^2x-a^3$.

189. Problems in Ratio and Proportion.

38. Find the ratio of two numbers whose sum is to their difference in the ratio $m : n$.

39. The speed of the steamship Servia is to that of the Bothnia as 13 to 10; and the first steams 5 miles farther in 8 hours than the second does in 10 hours. What is the speed of each?

40. The speed of two pedestrians was as 4 : 3, and the slower was 5 hours longer in going 36 miles than the slower was in going 24. What was the speed of each?

41. A chemist had two vessels, A, containing acid, and B, an equal quantity of water. He poured one third the acid into the water, and then poured one third of this mixture back into the acid. What was then the ratio of acid to water in A?

42. If 24 grains of gold and 400 grains of silver are each worth one dollar, what will be the weight of a coin containing equal parts of gold and silver and worth a dollar?

43. A chemist has two mixtures of alcohol and water, the one containing 90 per cent. of alcohol, the other 50 per cent. How much of the first must he add to 1 litre of the second to make a mixture containing 80 per cent. of alcohol.

44. It is a law of mechanics that the distances through which heavy bodies will fall in a vacuum in different times are proportional to the squares of the times. If a body fall 48 feet farther in 2 seconds than in 1 second, how far will it fall in 1 second? How far in t seconds?

45. On a line are two points whose distance is a. The first point divides the line into parts whose ratio is 2 : 3; the second into parts whose ratio is 5 : 7. What is the length of the line in terms of a?

46. If a line is divided into two parts whose ratio is $m : n$, what is the ratio of the length of the whole line to the distance of the point of division from the middle point?

47. A line is divided into three segments proportional to the numbers m, p and q. What is the ratio of the parts into which the middle point of the whole line divides the middle segment?

48. A sailing-ship leaves port, and 12 hours later is followed by a steamship. The ratio of the speeds being 3 : 8, how long will it take the steamer to overtake the ship?

49. A courier started from his post, going 7 miles in 3 hours. Two hours later another followed, going 7 miles in 2 hours. How long will the second be overtaking the first?

50. The areas of the openings of two water-faucets are in the ratio 3 : 5; the speeds of flow of the water through the openings are in the ratio 3 : 4. At the end of an hour 1221 gallons more have flowed through the second than through the first. What was the flew from each?

51. The speeds of two trains, A and B, are as m to n, and the journeys they have to make as $p : q$. It took train B t hours longer to make its journey than it did train A. What was the length of each journey and the speed of each train?

52. A street-railway runs along a regular incline, in consequence of which the speeds of the cars going in the two directions are as 2 : 3. The cars leave each terminus at regular intervals of 5 minutes. At what intervals of time will a car going up hill meet the successive cars coming down, and *vice versa?*

53. The same thing being supposed, two cars starting out simultaneously from the two ends of the route meet in 30 minutes. How long in time is the journey for each car?

54. Give the algebraic answers to the two preceding questions when the ratio of the speeds is $m : n$.

55. Find two numbers which are to each other as 4 : 3, and whose difference is $\frac{1}{3}$ of the less.

56. If $x : y :: 6 : 8$ and $4x - 3y = 7$, what is the value of x and y?

57. A year's profits were divided among two partners in the proportion of 3 : 4. If the second should give \$425 to the first, their shares would be equal. What was the amount divided?

58. In a first year's partnership A had 3 shares, and B 4. In the second A had 1, and B 2. In the second year A had \$300 less than he had the first, and B had \$200 more. What were the profits?

59. In a farm-yard there are 4 sheep to every 3 cattle, and

5 cattle to 6 hogs. How many hogs are there to every 20 sheep?

60. A drover started to market with a herd of 7 horses to every 5 mules. He sold 27 horses and bought 3 mules, and then had 3 horses to every 4 mules. How many of each had he at first?

61. Find two quantities whose sum, difference and product are proportional to 5, 1 and 12 respectively.

62. What number is that to which if 2, 6 and 10 be severally added, the first sum shall be to the second as the second is to the third?

63. What two numbers are to each other as 3 to 4, to each of which if 4 be added, the sums will be as 4 to 5?

64. What quantity must be taken from each term of the ratio $m : n$ that it may equal the ratio $c : d$?

65. If $a : b$ be the square of the ratio of $a + c : b + c$, show that c is a mean proportional between a and b.

66. If $a : b :: c : d$, show that $a(a + b + c + d) = (a + b)(a + c)$.

67. A line is divided by one point into two parts in the ratio of 3 : 5, and by another point into two parts in the ratio of 1 : 3. The distance between the points of division is 1 inch. What is the length of the line?

68. One ingot contains two parts of gold and one of silver, and another two parts of gold and three of silver. If equal parts are taken from each ingot, what will be the proportion of the gold to the silver in the alloy?

69. If two ounces be taken from the first of the above ingots and three from the second, what will be the ratio of the gold to the silver?

70. A cask contains 4 gallons of water and 18 gallons of alcohol. How many gallons of a mixture containing 2 parts water to 5 parts alcohol must be put in the cask so that there may be 2 parts water to 7 alcohol?

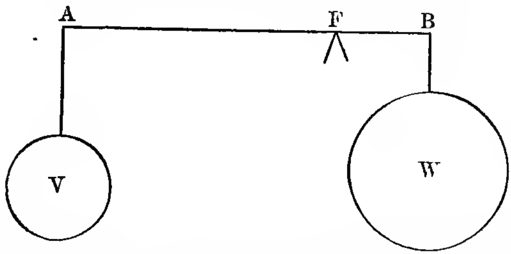
71. Which is the greater ratio, $1 + a : 1 - a$ or $1 + a^2 : 1 - a^3$, a being positive and less than 1?

72. Which is the greater ratio, $a^2 - ab + b^2 : a^2 + ab + b^2$ or $a^2 - a^2b^2 + b^4 : a^4 + a^2b^2 + b^4$, a and b having like signs?

73. What number must be taken from the second term of the ratio 2 : 34 and added to the first that it may equal 5 : 6?

74. It is a theorem of mechanics that, in order that two masses, V and W, at the ends of a lever, AB, may be in equilibrium, the distances of their points of suspension, A and B, from the fulcrum, F, must be *inversely proportional* to their weights; that is, we must have

$$\text{Weight V : weight W} = \text{FB : FA}.$$



Now, if the length AB of the lever is l, and the weights of V and W are respectively m and n, express the lengths AF and FB of the arms of the lever.

75. The weights at the ends of a lever are 8 and 13 kilogrammes, and the fulcrum is 3 inches from the middle of the lever. What is the length of the lever?

76. The sum of the two weights is 25 pounds, and the ratio of the distance of the fulcrum from the middle point to the length of the lever is 2 : 9. What are the weights?

77. The weights are m and n ($m > n$), and one arm of the lever is h longer than the other. Express the length of the lever.

78. A lever was balanced with weights of 7 and 9 kilogrammes at its ends. One kilogramme being taken from the lesser and added to the greater (making the weights 6 and 10 kilogrammes), the fulcrum had to be moved 2 inches. What was the length of the lever?

79. What number must be taken from each term of the ratio 19 : 30 that it may equal the ratio 1 : 2?

80. If $a : b :: c : d$, show that $a : b :: \sqrt{a^2 + c^2} : \sqrt{b^2 + d^2}$.

Find the ratio $x : y$ from the following proportions:

81. $x + y : x - y = m : n.$

82. $x + 2y : x - 2y = m : 2n.$

83. $x + 2y : 2x + y = m : n.$

84. $mx + ny : my + nx = p : q.$

85. $mx + ny : mx - ny = p : q.$

272. Quadratic Equations with Several Unknown Quantities.

14. $x^2 + y^2 = a;$ $x^2 - y^2 = b.$

15. $ax^2 + by^2 = c;$ $a'x^2 + b'y^2 = c'.$

16. $x^2 - y^2 = a;$ $xy = b.$

17. $x^3 + y^3 = 152;$ $xy = 15.$

18. $x^n + y^n = a;$ $xy = b.$

19. $x^2 + y^2 = a;$ $x + y = b.$

20. $x^2 - y^2 = m;$ $x^4 - y^4 = n.$

21. $x^2 + y^2 = h;$ $x - y = k.$

22. $\dfrac{x+y}{y} = mx;$ $xy = n.$

23. $\dfrac{x + \sqrt{1+x^2}}{y + \sqrt{1+y^2}} = a^2;$ $(x + \sqrt{1+x^2})(y + \sqrt{1+y^2}) = b^2.$

24. $x^2 + y^2 + x + y = m;$ $x^2 - y^2 + x - y = n.$

25. $x^2 + y^2 + x - y = a + b.$ $(x^2 + y^2)(x - y) = ab.$

26. $x^2 - xy = a^2y;$ $xy - y^2 = b^2x;$

27. $(x^2 - y^2)(x - y) = a;$ $(x^2 + y^2)(x + y) = b.$

28. $x^2 + y^2 + z^2 = 84;$ $x + y + z = 14;$ $xy = 8.$

29. $x + y + z = 12;$ $xy + yz + zx = 47;$ $x^2 + y^2 - z^2 = 0.$

30. $x + y = 9;$ $u + v = 9;$ $x^2 + u^2 = 52;$ $y^2 + v^2 = 41.$

31. $x + u = 13;$ $xy = 35;$ $y + v = 9;$ $uv = 18.$

32. The panel in a door is 12 by 18 inches, and it is to be surrounded by a margin of uniform width and equal surface to the panel. How wide must the margin be?

33. The fore wheel of a coach makes 6 more revolutions than the hind wheel in going 160 yards; but if the circumference of each wheel be increased by 4 feet, the fore wheel will make only 4 more revolutions in 160 yards. What is the circumference of each wheel?

34. The sum of three numbers is 15; the difference between the first and third is 3 more than the difference between the second and third, and the sum of their squares is 93. What are the numbers?

35. A principal of \$6000 amounts with simple interest to \$7800 after a certain number of years. Had the rate been 1 per cent. higher and the time 1 year longer, it would have amounted to \$720 more. What was the time and rate?

36. A courier left a town riding at a uniform rate. Three hours afterwards another followed, going 1 mile an hour faster. Two hours after the second another started, going 6 miles an hour. They arrive at their destination at the same time. What was the distance and rate of riding?

Ans. Dist. = 60 or 6. Speeds, 4, 5 and 6 or 1, 2 and 6.

37. In a right-angled triangle the hypothenuse is 5 and the area 6. What are the sides?

38. Find two numbers whose product is 180, and if the greater be diminished by 5 and the less increased by 3 the product of the sum and difference will be 150.

39. Find two numbers whose sum is 100 and the sum of their square roots 14.

40. Find two numbers whose sum is 35 and the sum of their cube roots 5.

41. By selling a horse for $130 I gain as much per cent. as the horse cost me. What did I pay for him?

42. What is the price of apples a dozen when four less in 20 cents' worth raises the price 5 cents per dozen?

43. The sum of the squares of three consecutive numbers is 149. What are the numbers?

44. If twice the product of two consecutive numbers be divided by three times their sum the quotient will be $\frac{8}{7}$. What are the numbers?

45. A woman bought a number of oranges for 36 cents. If she had bought 2 more for the same money she would have paid $\frac{1}{4}$ of a cent less for each orange. How many did she buy?

46. In mowing 60 acres of grass, 5 days less would have been sufficient if 2 acres more a day had been mown. How many acres were mown per day?

47. A broker bought a certain number of shares (par value $100 each) at a discount for $6400. When they were at the same per cent. premium, he sold all but 20 for $7200. How many shares did he buy, and at what price?

48. If the length and breadth of a rectangle were each increased by 2, the area would be 238; if both were each diminished by 2, the area would be 130. Find the length and breadth.

49. Twice the product of two digits is equal to the number itself; and 7 times the sum of the digits is equal to the number formed by the same digits reversed. What is the number?

50. The sum of two numbers is $\frac{5}{3}$ of the greater, and the difference of their squares is 45. What are the numbers?

51. The numerator and denominator of two fractions are each greater by 2 than those of another; and the sum of the two fractions is $2\frac{5}{6}$; if the denominators were interchanged, the sum of the two fractions would be 3. What are the fractions?

52. A man starts from A to go to B. During the first half of the journey he drives $\frac{1}{2}$ mile an hour slower than the other half, and arrives in $5\frac{2}{3}$ hours. On his return he travels a mile slower during the first half than when he went in going over the same portion, and returned in $6\frac{3}{7}$ hours. What was the distance and rate of driving?

53. A person who has \$8800 invests a part of it in one enterprise and the rest in another; the dividends differ in rate, but are equal in amount. If the sums invested had exchanged rates of dividends, the first would have yielded \$200 and the other \$288. What were the rates?

54. Divide 50 into two such parts that their product may be to the sum of their squares as 6 to 13.

55. A company at a hotel had \$12 to pay, but before settling 2 left, when those remaining had 30 cents apiece more to pay than before. How many were there?

56. A and B set out from two towns which are 144 miles apart, and travelled until they met. A went 8 miles an hour, and the number of hours they travelled was three times greater than the number of miles B travelled an hour. At what time did they meet, and what was B's speed?

57. In a purse containing 28 pieces of silver and nickel, each silver coin is worth as many cents as there are nickel coins, and each nickel is worth as many cents as there are silver coins, and the whole are worth \$1.50. How many are there of each?

58. Find two such numbers that the product of their sum and difference may be 7, and the product of the sum and difference of their squares may be 144.

59. A grocer sold 50 pounds of pepper and 80 pounds of ginger for \$26; but he sold 25 pounds more of pepper for \$10 than he did of ginger for \$4. What was the price per pound of each?

285. Arithmetical Progression.

23. Find the nth term of the series 1, 3, 5, 7, etc.

24. Find the sum of n terms of the series 2, 4, 6, 8, etc.

25. Find the sum of n terms of the series 1, 3, 5, 7, etc.

26. Insert six arithmetical means between 2 and 23.

27. If a body fall 16 feet during the first second, and 32 feet each second thereafter more than in the immediately preceding second, how far will it fall during the tenth second, and how far in ten seconds?

28. Find the sum of n terms of the series 5, 12, 19, 26, 33, etc.

29. Find the sum of 12 terms of the series 7, $\frac{13}{2}$, 6, etc.

30. What is the expression for the sum of n terms of a series whose first term is $\frac{3}{2}$ and the difference between the third and seventh terms 3?

31. The difference between the first and tenth term of an increasing A. P. is 18, and the sum of the ten terms is 100. What is the A. P.?

32. $a = 3$, and the fourth term is 4. What is the sum of eight terms?

33. There are two arithmetical series which have the same common difference; the first terms are 2 and 3 respectively, and the sum of five terms of one is to the sum of five terms of the other as 8 : 9. What are the series?

34. There are four numbers in A. P. whose sum is to the sum of their squares as 2 : 12, and the sum of the first three is 12. What is the A. P.?

35. A number consisting of three digits, which are in arithmetical progression, if divided by the sum of its digits gives 15 for a quotient, and if 396 be added to it the digits will be reversed. What is the nnmber?

36. Find three numbers in arithmetical progression the sum of whose squares shall be 261, and the square of the mean greater than the product of the extremes by 9.

37. Find four numbers in arithmetical progression whose sum is 16 and continued product 105.

38. A starts from a certain place going 2 miles the first day, 4 the second, 6 the third, and so on; three days after B sets out and travels 15 miles a day. How many days before B overtakes A?

39. A traveller started from a certain place going 2 miles the first day, 5 the second, and so on. After two days another followed and went 6 miles the first day, 10 the second, and so on. After how many days will they be together?

40. In a series of five terms the sum of the first three is 24, and the last three 42. What is the series?

41. Find three numbers in arithmetical progression whose sum shall be 36, and the sum of the first and second shall be $\frac{5}{7}$ of the sum of the second and third.

42. How many means must be inserted between 5 and 17 in order that the sum of the first two shall be to the sum of the second two as 4 : 7?

43. In an arithmetical progression of three terms whose common difference is 4, the product of the second and third is greater by 12 than the product of the first and second. What is the series?

44. The sum of the squares of the extremes of an arithmetical progression of four terms is 153, and the sum of the squares of the means 117. What is the series?

45. The sum of five terms of an arithmetical progression is 50, and the product of the first and fifth is to the product of the second and fourth as 7 : 8. What is the series?

46. The sum of nine terms of an arithmetical progression is 90. What is the middle term, and the sum of the extremes?

47. A and B start together from the same place; A goes 20 miles a day and B 15 miles the first day and $\frac{1}{2}$ mile more on each succeeding day than on the preceding day. How far apart will they be at the end of 10 days, and which will be in advance?

48. A and B have a distance of 27 miles to walk; A starts at $2\frac{1}{2}$ miles an hour with an hourly increase of $\frac{1}{4}$ mile; B starts at 5 miles an hour, but falls off $\frac{1}{2}$ mile every hour. Which will finish first, and by how much?

49. Find a progression of four terms in which the sum of the extremes is 21 and the product of the means 104.

292. Geometrical Progression.

21. Find three numbers in A. P. which being increased by 1, 3, and 10 respectively the sum will form a geometrical progression whose common ratio is 2.

22. Find three numbers in geometrical progression whose sum is 19 and the sum of whose reciprocals $\frac{19}{36}$.

23. Express the sum of n terms of the G. P. whose first term is a and common ratio b^2.

24. Express the sum of n terms when the second term is 2 and the third is -6.

25. Show that the sum of $2n$ terms when divided by the sum of n terms gives the quotient $r^n + 1$.

26. Find five terms in a geometrical progression the sum of whose three means is 156 and the sum of the two extremes 328.

27. In a geometrical progression of three terms, the sum of the first and second exceeds the third by 2, and the sum of the first and third exceeds the second by 14. What is the series?

28. The sum of two numbers is 30, and the arithmetical mean exceeds the geometrical by 3. What are the numbers?

29. Having a progression of an odd number of terms, show that if from the sum of the even terms (the second, fourth, etc.) we subtract the sum of all the odd terms except the last (the first, third, etc.), the remainder is equal to the difference between the first and last terms divided by $r + 1$.

30. In a geometrical progression of four terms the first is less than the third by 24, and the sum of the extremes is to the sum of the means as $7 : 3$. What is the series?

31. The sum of three numbers in geometrical progression is 38, and the product of the mean by the sum of extremes is 312. What is the series?

32. Find a geometrical progression of three terms whose product is 1728 and the difference of extremes 45.

33. The sum of four terms of a geometrical progression is 80, and the last term divided by the sum of the two means is $2\frac{1}{4}$. What is the series?

34. If 1, 3 and 7 be added to three consecutive terms of an arithmetical progression whose C. R. is 2, the sums will form a geometrical progression. Find the series?

35. Find four numbers in a geometrical progression such that the fourth shall be 144 more than the second, and the sum of the means be to the sum of the extremes as 3 : 7.

36. Find a geometrical progression in which the sum of the second and third terms is 60, and of the first and third 50.

37. The sum of the first and second of four terms of a geometrical progression is 16, and the sum of the third and fourth terms 144. What is the series?

38. Show that the difference of any two terms of a G. P. is divisible by $r - 1$. (Cf. § 136.)

39. Find a geometrical progression of three terms whose product is 729, and the sum of the extremes divided by the means is $3\frac{1}{3}$.

40. The difference of two numbers is 48, and the arithmetic mean exceeds the geometric by 18. What are the numbers?

41. The fifth term of a G. P. exceeds the first by 16, and the fourth exceeds the second by $4\sqrt{3}$. Find the first term and common ratio.

42. In a G. P. the sum of n terms is S, and the sum of $2n$ terms in $6S$. Express the common ratio and first term.

43. In a G. P. of $2n + 1$ terms, whose first term is 5, the sum of the first and last terms is 125 greater than twice the middle term. Find the common ratio.

44. The first term of a G. P. is 2, and the continued product of the first five terms is 128. What is the common ratio?

45. Find that G. P. of which the product of the first and second terms is 3, and that of the third and fourth terms 48.

46. A person who each year gained half as much again as he did the year before, gained \$2059 in 7 years. What was his gain the first year?

THE END.

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